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Hilbert polynomials for the extension functor

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Abstract

Let R be a local ring, $I \subseteq R$ an ideal, and M and N finite R-modules. In this paper we provide a number of results concerning the degree of the polynomial giving the lengths of the modules $\operatorname{Ext}^i_R(N/I^nN, M)$, when such a polynomial exists. Included among these results are a characterization of when this degree equals the Krull dimension of R, a characterization of when the degree of the polynomial associated to the first non-vanishing Ext under consideration equals the grade of I on M, and calculation of the degree of Hilbert polynomials associated to certain iterated expressions involving the extension functor.

Keywords: Hilbert–Samuel polynomial; Extension functor; Injective resolution

1. Introduction

In this paper we continue our investigation into the degrees of Hilbert polynomials associated to derived functors, in this case focusing on the extension functor. Let $(R, \mathfrak{m}, k)$ be a local ring and M, N be finite R-modules. Let $I \subseteq R$ be an ideal such that $I + \operatorname{ann}(M) + \operatorname{ann}(N)$ is $\mathfrak{m}$-primary. For a finite length R-module A, write $\lambda(A)$ for the length of A. It is shown in [6] that

$$ {}^{i}_{I}(n) := \lambda\big(\operatorname{Ext}^i\big(N/I^nN, M\big)\big) $$

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has polynomial growth for n large. Moreover, a degree estimate for this polynomial is given in terms of the dimension of the Matlis dual of cohomology modules derived from an injective resolution of M. Here we seek to give cases where this degree and the corresponding normalized leading coefficient can be explicitly computed. One of our main results, Theorem 3.2, characterizes when the degree of $\ ^{i}_{I}(n)$ equals d, the dimension of R. It turns out that if N is locally free at primes of maximal dimension (e.g., $N = R$), this can only occur when $i = d$. This is a satisfactory finding since the analogous polynomial associated to the torsion functor can only achieve degree d when the Tor index is 0. Moreover, in the case $N = R$, we show that the normalized leading coefficient of $\ ^{d}_{I}(n)$ is $e(I, M)$, the multiplicity of I on M. In [5] Kirby gave an early result concerning the behavior of $\ ^{g}_{I}(n)$, where g denotes the grade of I on M, noting that the associated polynomial has degree less than or equal to g. We go further in Theorem 3.9, where we not only give an explicit description of the lengths determining $\ ^{g}_{I}(n)$, but we also give precise conditions for the degree to equal g. In Theorem 3.11 we exhibit several classes of ideals where for $i = d$, $\ ^{i}_{I}(n)$ has the expected maximal degree $d - 1$, in light of Theorem 3.2. Recall that if R is Gorenstein, then for any $\mathfrak{m}$-primary ideal I, R/I^n and $\mathrm{Ext}^d(\mathrm{Ext}^d(R/I^n, R), R)$ are isomorphic and therefore give rise to the same Hilbert polynomials. In section four we give the degree and leading coefficient of Hilbert polynomials derived from similar iterated extension modules. In particular, we show that if $I \subseteq R$ is an $\mathfrak{m}$-primary ideal in any local ring, then the polynomials giving the lengths of R/I^n and $\mathrm{Ext}^d(\mathrm{Ext}^d(R/I^n, R)R)$ have the same degree and same normalized leading coefficient.

2. Preliminaries

Throughout we assume that R is a local Noetherian ring with maximal ideal $\mathfrak{m}$, residue field k and Krull dimension d. We also assume throughout that $d > 0$, since in the case where $\dim(R) = 0$, the Hilbert polynomials under consideration are constants (often identically zero). We will rely on standard facts from Hilbert–Samuel theory. Namely, that if U is a finitely generated R-module and $I \subseteq R$ is an ideal such that $\lambda(U/IU) < \infty$, then the lengths of the modules U/I^nU are finite and given by a polynomial $P(n)$ with rational coefficients for n large. As is well known, the degree of $P(n)$ equals the dimension of U and the normalized leading coefficient of $P(n)$, denoted $e(I, U)$, is called the multiplicity of I on U. More generally, let $\{H_n\}_{n \geqslant 0}$ be any family of finite length modules with the property that there exists a rational polynomial $Q(n)$ such that $\lambda(H_n) = Q(n)$ for n large. Then the *normalized leading coefficient* of $Q(n)$ is the positive integer

$$\lim_{n \to \infty} \frac{\deg(Q(n))!}{n^{\deg(Q(n))}} \cdot Q(n).$$

For a finitely generated R-module V and an ideal $I \subseteq R$, we write $\ _V(I)$ for the *analytic spread* of I on V, i.e., the Krull dimension of the graded module $\bigoplus_{n \geqslant 0} I^nV/\mathfrak{m}I^nV$. It is well known that $\ _V(I) = \ _{R/\mathrm{ann}(V)}(I)$, the analytic spread of the image of the ideal in the ring $R/\mathrm{ann}(V)$. A proof of this can be found in the proof of Proposition 3 in [6]. In particular, $\ _V(I) \leqslant \dim(V)$.

For finitely generated R-modules M and N and an ideal $I \subseteq R$, the main result in [6] shows that if $\mathrm{ann}(M) + \mathrm{ann}(N) + I$ is $\mathfrak{m}$-primary, the lengths of $\mathrm{Ext}^i(N/I^nN, M)$ are given by a polynomial in n, for n large. For the purposes of this paper, we make the following definition.

Definition 2.1.

(a) Let M and N be finitely generated modules over R and $I \subseteq R$ an ideal. We shall say that M, N, and I satisfy the *standard support condition* if $I + \operatorname{ann}(M) + \operatorname{ann}(N)$ is $\mathfrak{m}$-primary.
(b) If the support condition in (a) holds, we will write ${}^{i}_{I}(n)$ for the polynomial giving the lengths of $\operatorname{Ext}^i(N/I^nN, M)$ for n large.

We will also use the following notation.

Definition 2.2. Let $j \geqslant 0$ be an integer and M a finitely generated R-module having an associated prime of dimension j. Let $P_1, \ldots, P_r$ be the prime ideals in $\operatorname{Ass}(M)$ of dimension j and $J = P_1 \cap \cdots \cap P_r$. We define M_j to be the set of elements in M annihilated by some power of J, i.e., $M_j := \Gamma_J(M)$. Note that for any $1 \leqslant i \leqslant r$, $(M_j)_{P_i} = \Gamma_{P_i}(M_{P_i})$.

We will make use of the following facts about Ext modules and injective resolutions. These facts will be used in the sequel with little or no further comment. First, suppose M is a finite R-module and S is a Gorenstein local mapping onto R. If $j := \operatorname{depth}(S) - \operatorname{depth}(M)$, then j is the largest index for which $\operatorname{Ext}^j_S(M, S) = 0$ (see [1, 8.1.8 and 8.1.9]). Now, let $\mathcal{E}$ be a minimal injective resolution of M. The jth Bass number $\mu^j := \mu^j(\mathfrak{m}, M)$ of M with respect to $\mathfrak{m}$ is $\dim_k(\operatorname{Ext}^j(k, M))$, so that μ^j is just the number of times the injective hull of k appears as a summand of the jth injective module in $\mathcal{E}$. It is well known that $\mu^j = 0$, for $j < \operatorname{depth}(M)$ and $\mu^j = 0$ for $j = \operatorname{depth}(M)$. Now suppose that N is a finitely generated R-module and I is an ideal such that N, M and I satisfy our standard support condition. In [6, p. 84], it is shown that

$$\operatorname{Hom}\bigl(N/I^nN, \mathcal{E}\bigr) = \operatorname{Hom}\bigl(N/I^nN, \Gamma_{\mathfrak{m}}(\mathcal{E})\bigr), \tag{2.1}$$

where $\Gamma_{\mathfrak{m}}(-)$ is the local cohomology functor. In particular, this means that $\operatorname{Ext}^j(N/I^nN, M)$ is the jth cohomology of the complex $\operatorname{Hom}(N/I^nN, \Gamma_{\mathfrak{m}}(\mathcal{E}))$. In other words,

$$\operatorname{Ext}^j\bigl(N/I^nN, M\bigr) = H^j\bigl(\operatorname{Hom}\bigl(N/I^nN, \Gamma_{\mathfrak{m}}(\mathcal{E})\bigr)\bigr). \tag{2.2}$$

Finally, for a complex of R-modules $\mathcal{C}$, we will denote the Matlis dual of $\mathcal{C}$ by $\mathcal{C}^\vee$.

Before starting, we need a lemma which will help us to estimate or calculate the normalized leading coefficient of ${}^{i}_{I}(n)$. This lemma leads to an improved statement regarding the degree estimate given in [6, Corollary 7]. Suppose that U, V, W are submodules of a common finitely generated R-module with $W \subseteq V$. Let $I \subseteq R$ be an ideal such that the modules $L_n := (U + I^nV)/I^nW$ have finite length for n large. Then, it follows from Lemma 2 in [6] that these lengths are given by a polynomial in n for n large. We will use this fact in the lemma below.

Lemma 2.3. *Suppose that U, V, W are submodules of a finitely generated R-module with $W \subseteq V$. Let $I \subseteq R$ be an ideal such that the modules $L_n := (U + I^nV)/I^nW$ have finite length for n large. Let $P(n)$ denote the corresponding Hilbert polynomial, i.e., $P(n) = \lambda(L_n)$, for n large.*

(i) *If* $\dim(U) \geqslant {}_V(I) - 1$, *then the degree of $P(n)$ equals* $\dim(U)$ *and the normalized leading coefficient of $P(n)$ is at least $e(I, U)$.*
(ii) *If* $\dim(U) \geqslant {}_V(I)$, *then the normalized leading coefficient of $P(n)$ equals $e(I, U)$.*

Proof. Consider the canonical short exact sequence where π is the sum map

$$0 \to \frac{U \cap I^n V}{U \cap I^n W} \to \frac{U + I^n W}{I^n W} \oplus \frac{I^n V}{I^n W} \xrightarrow{\pi} L_n \to 0.$$

Note that all of the terms in this sequence have finite length, so it follows that for n large,

$$P(n) = \lambda\big((U + I^n W)/I^n W\big) + \lambda\big(I^n V/I^n W\big) - \lambda\big((U \cap I^n V)/U \cap I^n W\big). \tag{2.3}$$

The second and third terms in this equation are given by polynomials, since the graded modules

$$\bigoplus_{n\geqslant 0} I^n V/I^n W \quad \text{and} \quad \bigoplus_{n\geqslant 0} \big(U \cap I^n V\big)/U \cap I^n W$$

are finitely generated over the Rees ring of R with respect to I. Moreover, the dimensions of these modules are bounded by $_V(I)$. Thus the degrees of the polynomials giving the second and third terms in Eq. (2.3) are bounded by $_V(I) - 1$. Now, since

$$\lambda\big(I^n V/I^n W\big) - \lambda\big((U \cap I^n V)/U \cap I^n W\big) \geqslant 0$$

the degree and normalized leading coefficient of $P(n)$ are at least the degree and normalized leading coefficient of the polynomial giving the lengths of $(U + I^n W)/I^n W$, provided this polynomial has degree at least $_V(I) - 1$. Write $Q(n)$ for this latter polynomial. We claim that the degree and normalized leading coefficient of the polynomial of $Q(n)$ are $\dim(U)$ and $e(I, U)$ respectively. If we show this, then the conclusions of the lemma will follow.

Using Artin–Rees, write

$$\big(U + I^n W\big)/I^n W \cong U/\big(U \cap I^n W\big) = U/I^{n-t}\big(U \cap I^t W\big),$$

for some $t \geqslant 0$ and all $n \geqslant t$. This shows that $e(I, U)$ is defined, and that the claim is true if $\dim U = 0$. Assuming $\dim U > 0$, write

$$\lambda\big((U + I^n W)/I^n W\big) = \lambda\big(U/(U \cap I^t W)\big) + \lambda\big((U \cap I^t W)/I^{n-t}(U \cap I^t W)\big).$$

Thus, the degree of $Q(n)$ is the dimension of $U \cap I^t W$ and the normalized leading coefficient of $Q(n)$ is $e(I, U \cap I^t W)$, assuming $\dim(U \cap I^t W) > 0$. But now, since the quotient $U/U \cap I^t W$ has finite length, the modules U and $U \cap I^t W$ have the same support. Thus, $\dim(U \cap I^t W) = \dim(U) > 0$. In particular, for a prime $P \subseteq R$, P is a prime of maximal dimension in the support of $U \cap I^t W$ if and only if P is a prime of maximal dimension in the support of U. If we now apply the associativity formula, we get that the multiplicity of I on both U and $U \cap I^t W$ is the same:

$$\begin{aligned} e\big(I, U \cap I^t W\big) &= \sum_{\dim P = \dim(U \cap I^t W)} \lambda\big(U \cap I^t W\big)_P e(I, R/P) \\ &= \sum_{\dim P = \dim U} \lambda(U_P) e(I, R/P) \\ &= e(I, U), \end{aligned}$$

which completes the proof of the lemma. □

Remark 2.4. Before continuing, wish to give an application of Lemma 2.3 by improving [6, Corollary 7]. While an estimate for the degree of ${}^{i}_{I}(n)$ was given in [6], there were no statements regarding criteria for equality in that degree estimate nor were there any statements concerning the normalized leading coefficient of ${}^{i}_{I}(n)$. The following proposition, used throughout this paper, remedies this.

Proposition 2.5. *Let R be a local ring, $I \subseteq R$ an ideal and let M and N be finitely generated R-modules. Assume that our standard support condition holds. Let $\mathcal{E}$ be a minimal injective resolution of M. Then*

$$\deg\bigl({}^{i}_{I}(n)\bigr) \leqslant \max\bigl\{\dim\bigl(\mathrm{H}_i\bigl(\mathrm{Hom}\bigl(N, \Gamma_{\mathfrak{m}}(\mathcal{E})\bigr)^{\vee}\bigr)\bigr), \ell_N(I) - 1\bigr\}.$$

Furthermore,

(i) *The inequality in the estimate above becomes an equality whenever the dimension of $\mathrm{H}_i(\mathrm{Hom}(N, \Gamma_{\mathfrak{m}}(\mathcal{E}))^{\vee})$ is greater than or equal to ${}_N(I) - 1$.*
(ii) *If $\dim(\mathrm{H}_i(\mathrm{Hom}(N, \Gamma_{\mathfrak{m}}(\mathcal{E}))^{\vee}))$ is greater than or equal to ${}_N(I) - 1$, the normalized leading coefficient of ${}^{i}_{I}(n)$ is at least $e(I, \mathrm{H}_i(\mathrm{Hom}(N, \Gamma_{\mathfrak{m}}(\mathcal{E}))^{\vee}))$.*
(iii) *If $\dim(\mathrm{H}_i(\mathrm{Hom}(N, \Gamma_{\mathfrak{m}}(\mathcal{E}))^{\vee}))$ is greater than or equal to ${}_N(I)$, then the normalized leading coefficient of ${}^{i}_{I}(n)$ equals $e(I, \mathrm{H}_i(\mathrm{Hom}(N, \Gamma_{\mathfrak{m}}(\mathcal{E}))^{\vee}))$.*

Proof. The displayed inequality has already been given in [6, Corollary 7]. For the remaining parts of the proposition, we just need to translate between the notation in [6] and the notation of the lemma. First note that Eq. (10) in [6] shows that the Matlis dual of $\mathrm{Ext}^i(N/I^nN, M)$ is isomorphic to the ith homology of the complex $\mathcal{C} \otimes R/I^n$, where $\mathcal{C} := \mathrm{Hom}(N, \Gamma_{\mathfrak{m}}(\mathcal{E}))^{\vee}$ is a complex whose modules are finite direct sums of N. The proof of [6], Proposition 3(a) shows that the homology of a complex of the form $\mathcal{C} \otimes R/I^n$ can be written as

$$\mathrm{H}_i\bigl(\mathcal{C} \otimes R/I^n\bigr) = \frac{K + I^{n-n_0}\tilde{K}}{L + I^{n-n_0}C},$$

with $L \subseteq K$ and $C \subseteq \tilde{K}$. This quotient can be written as $(U + I^{n-n_0}V)/I^{n-n_0}W$, where $U := K/L$ is the ith homology of $\mathcal{C}$, $V := (\tilde{K} + L)/L$ is a subquotient of a direct sum of finitely many copies of N and $W := (C + L)/L$ (see [6, p. 81, lines 8 and 9]). We now have the form required by Lemma 2.3; in other words, for n large, ${}^{i}_{I}(n)$ gives the lengths of $(U + I^{n-n_0}V)/I^{n-n_0}W$. Once we observe that ${}_V(I) \leqslant {}_N(I)$, the remaining statements in the proposition will follow immediately from the lemma. However, ${}_V(I) = {}_{R/\operatorname{ann}(V)}(I)$ and ${}_N(I) = {}_{R/\operatorname{ann}(N)}(I)$. Since V is a subquotient of a finite direct sum of N, $\operatorname{ann}(N) \subseteq \operatorname{ann}(V)$, so ${}_{R/\operatorname{ann}(V)}(I)$ is less than or equal to ${}_{R/\operatorname{ann}(N)}(I)$, which gives what we want. □

3. Degree and leading coefficient of ${}^{i}_{I}(n)$

In this section we present our results concerning the degree and leading coefficient of ${}^{i}_{I}(n)$. Because the estimate in Proposition 2.5 involves two terms, and equality holds in this estimate when the first of these terms dominates, our best results, with I as general as possible, occur when the degree in question equals d or $d - 1$. For completely different reasons we are able to show $\deg(\epsilon^{g}_{I}(n)) \leqslant g$, for $g := \mathrm{grade}_I(M)$, and give a criterion for equality.

We begin with the case that the degree of $\ {}_I^i(n)$ equals d. We isolate a crucial part of the argument for this case in the following lemma.

Lemma 3.1. *Let R be a complete local ring of dimension d, $I \subseteq R$ an ideal and M, N finite R-modules such that I, M, N satisfy our standard support condition. Let $\mathcal{E}$ denote a minimal injective resolution of M. Then the following are equivalent*:

(a) $\dim(\mathrm{H}_i(\Gamma_{\mathfrak{m}}(\mathcal{E})^\vee \otimes_R N)) = d$.
(b) $i \geqslant d$ *and* $\dim(\mathrm{Ext}^{i-d}(N, M)) = d$.

Proof. By Cohen's structure theorem, there exists a Gorenstein local ring $(S, \mathfrak{n})$ of dimension d that maps onto R. Thus, by local duality and the permanence of local cohomology, we have

$$\mathrm{H}^j_{\mathfrak{m}}(M)^\vee = \mathrm{H}^j_{\mathfrak{n}}(M)^\vee = \mathrm{Ext}^{d-j}_S(M, S),$$

for $0 \leqslant j \leqslant d$.

Let $P \subseteq R$ be any prime of dimension d and let $Q \subseteq S$ be its pre-image in S. We first note that since the modules in the complex $\Gamma_{\mathfrak{m}}(\mathcal{E})$ are finite sums of the injective hull of k, the modules in $\Gamma_{\mathfrak{m}}(\mathcal{E})^\vee$ are finitely generated free R-modules. Moreover, by construction, we have for all $i \geqslant 0$, $\mathrm{H}_i(\Gamma_{\mathfrak{m}}(\mathcal{E})^\vee) = \mathrm{Ext}^{d-i}_S(M, S)$ (where we take $\mathrm{Ext}^{d-i}_S(M, S) = 0$, for $i > d$). Thus,

$$\mathrm{H}_i\big(\Gamma_{\mathfrak{m}}(\mathcal{E})^\vee\big)_P = \mathrm{Ext}^{d-i}_S(M, S)_P = \mathrm{Ext}^{d-i}_S(M, S)_Q = 0,$$

for all $0 \leqslant i < d$, since S_Q is self-injective. Thus, the complex $(\Gamma_{\mathfrak{m}}(\mathcal{E})^\vee)_P$ is split exact in degrees i for $0 \leqslant i < d$. Thus, $(\Gamma_{\mathfrak{m}}(\mathcal{E})^\vee \otimes N)_P$ is split exact in degrees i, for $0 \leqslant i < d$. On the one hand, this immediately shows that $\mathrm{H}_i(\Gamma_{\mathfrak{m}}(\mathcal{E})^\vee \otimes N)$ has dimension less than d for all $0 \leqslant i < d$. On the other hand, if we split off the terms up to degree d from $(\Gamma_{\mathfrak{m}}(\mathcal{E})^\vee)_P$, then we obtain a free resolution of $\mathrm{Hom}_S(M, S)_P$ over R_P. Note that this latter module is just $M_P{}^{\vee_P}$, where $M_P{}^{\vee_P}$ denotes the Matlis dual of M_P over R_P. To see this, first observe $\mathrm{Hom}_S(M, S) = \mathrm{Hom}_R(M, \mathrm{Hom}_S(R, S))$, since S maps onto R; at the same time, $\mathrm{Hom}_S(R, S)_Q = \mathrm{Hom}_S(R, S)_P$ is the injective hull of R_P, since S_Q is Gorenstein. Putting these together yields $\mathrm{Hom}_S(M, S)_P = M_P{}^{\vee_P}$. Now, using exactness of Matlis duality, we have for $i \geqslant d$ that

$$\mathrm{H}_i\big(\Gamma_{\mathfrak{m}}(\mathcal{E})^\vee \otimes N\big)_P = \mathrm{H}_i\big(\big(\Gamma_{\mathfrak{m}}(\mathcal{E})^\vee\big)_P \otimes N_P\big) = \mathrm{Tor}^{R_P}_{i-d}\big(M_P^{\vee_P}, N_P\big).$$

Since the Matlis dual over R_P of the latter Tor module is $\mathrm{Ext}^{i-d}_{R_P}(N_P, M_P)$, we have that,

$$\lambda\big(\big(\mathrm{H}_i\big(\Gamma_{\mathfrak{m}}(\mathcal{E})^\vee \otimes_R N\big)\big)_P\big) = \lambda\big(\big(\mathrm{Ext}^{i-d}_R(N, M)\big)_P\big)$$

for all primes $P \subseteq R$ of dimension d. Thus for $i \geqslant d$, $\dim(\mathrm{H}_i(\Gamma_{\mathfrak{m}}(\mathcal{E})^\vee \otimes_R N)) = d$ if and only if $\dim(\mathrm{Ext}^{i-d}(N, M)) = d$. This completes the proof of the lemma. □

Theorem 3.2. *Let R be a local ring, $I \subseteq R$ an ideal and M, N finite R-modules such that I, M, N satisfy our standard support condition. Then, the following are equivalent*:

(a) $\deg(\epsilon^i_I(n)) = d$.
(b) $i \geqslant d$ *and* $\dim(\mathrm{Ext}^{i-d}(N, M)) = d$.

Moreover, if (a) *and* (b) *hold, then*

$$ {}^{i}_{I}(n) = \frac{e(I, \operatorname{Ext}^{i-d}(N, M))}{d!} n^d + \textit{lower degree terms}. $$

Proof. We first note that we are free to pass to the completion $\hat{R}$ of R and assume that R is complete.

Now, let $\mathcal{E}$ be a minimal injective resolution of M over R. It follows from Proposition 2.5 that

$$ \deg\bigl({}^{i}_{I}(n)\bigr) \leqslant \max\bigl\{\dim \mathrm{H}_i\bigl(\operatorname{Hom}\bigl(N, \Gamma_{\mathfrak{m}}(\mathcal{E})\bigr)^{\vee}\bigr), \ell_N(I) - 1\bigr\} $$

which by Hom-tensor duality gives

$$ \deg\bigl({}^{i}_{I}(n)\bigr) \leqslant \max\bigl\{\dim \mathrm{H}_i\bigl(\Gamma_{\mathfrak{m}}(\mathcal{E})^{\vee} \otimes_R N\bigr), \ell_N(I) - 1\bigr\}. \tag{3.1} $$

It also follows from Proposition 2.5 that equality holds in this degree estimate if the first term on the right is at least as large as the second. Thus, since ${}_N(I) \leqslant d$, $\deg(\epsilon^i_I(n)) = d$ if and only if $\dim(\mathrm{H}_i(\Gamma_{\mathfrak{m}}(\mathcal{E})^{\vee} \otimes_R N)) = d$. Thus, (a) and (b) are equivalent by Lemma 3.1.

Assume now that (a) and (b) are satisfied. Our standard support condition implies that $\lambda(\operatorname{Ext}^j(N, M)/I \operatorname{Ext}^j(N, M))$ is finite, so $e(I, \operatorname{Ext}^j(N, M))$ is defined for all $j \geqslant 0$. By Proposition 2.5, the normalized leading coefficient of ${}^{i}_{I}(n)$ is $e(I, \mathrm{H}_i(\Gamma_{\mathfrak{m}}(\mathcal{E})^{\vee} \otimes N))$. Using the associativity formula, we have the set of equalities

$$ \begin{aligned} e\bigl(I, \mathrm{H}_i\bigl(\Gamma_{\mathfrak{m}}(\mathcal{E})^{\vee} \otimes_R N\bigr)\bigr) &= \sum_{\dim P = d} \lambda\bigl(\mathrm{H}_i\bigl(\Gamma_{\mathfrak{m}}\bigl((\mathcal{E})^{\vee}\bigr) \otimes_R N\bigr)_P\bigr) e(I, R/P) \\ &= \sum_{\dim P = d} \lambda\bigl(\operatorname{Ext}^{i-d}(N, M)_P\bigr) e(I, R/P) \\ &= e\bigl(I, \operatorname{Ext}^{i-d}(N, M)\bigr), \end{aligned} $$

which completes the proof of the theorem. □

Example 3.3. Suppose that R has a prime P of dimension d such that R_P is not Gorenstein. Let $M := R$, $N := R/P$ and I be any $\mathfrak{m}$-primary ideal. Then it follows that for all $i \geqslant d$, $\operatorname{Ext}^{i-d}(N, M)_P = 0$, so $\operatorname{Ext}^{i-d}(N, M)$ has dimension d. Thus, by Theorem 3.2, $\deg(\epsilon^i_I(n)) = d$, for all $i \geqslant d$. On the other hand, Theorem 3.2 also shows that $\deg(\epsilon^i_I(n)) < d$, for all $i < d$.

We now collect some corollaries of both the proof and the statement of the theorem. Included among these results are the case N is locally free at all primes of maximal dimension. Note that this case occurs if $N = R$, N is a syzygy of a module that is free at primes of maximal dimension, or if N is a module with a rank (i.e., N_P is free of constant rank at each associated prime P of R).

Corollary 3.4. *Let R be a local ring, $I \subseteq R$ an ideal and M, N finite R-modules such that I, M, N satisfy our standard support condition. Assume further that N is free of rank $r > 0$ at all minimal primes of R of dimension d. Then, the following are equivalent*:

(a) $\deg(\epsilon^i_I(n)) = d$.
(b) $i = d$ *and there exists* $P \in \operatorname{Spec}(R)$ *such that* $\operatorname{ann}(M) + \operatorname{ann}(N) \subseteq P$ *and* $\dim(R/P) = d$.

Moreover, if (a) *and* (b) *hold and* $\operatorname{ann}(M) + I$ *is* $\mathfrak{m}$*-primary, then*

$$ {}^{d}_{I}(n) = \frac{r \cdot e(I, M)}{d!} n^d + \textit{lower degree terms}. $$

Proof. The equivalence of (a) and (b) follows immediately from the theorem. For the second statement, note that $e(I, M)$ is defined, since $\operatorname{ann}(M) + I$ is $\mathfrak{m}$-primary. By Theorem 3.2, the normalized leading coefficient of ${}^{d}_{I}(n)$ is $e(I, \operatorname{Hom}(N, M))$. The associativity formula and the fact that N_P is free of rank r at primes of maximal dimension yields

$$ \begin{aligned} e\big(I, \operatorname{Hom}(N, M)\big) &= \sum_{\dim P = d} \lambda\big(\operatorname{Hom}(N, M)_P\big) e(I, R/P) \\ &= \sum_{\dim P = d} r \cdot \lambda(M_P) e(I, R/P) \\ &= r \cdot e(I, M). \qquad \square \end{aligned} $$

In the next two corollaries we take $N = R$. We consider this to be an important case, since the local cohomology module $\mathrm{H}^i_I(M)$ is the direct limit of the modules $\operatorname{Ext}^i(R/I^n, M)$. We note also that the degree statement in part (b) of Corollary 3.5 was already known when the ring R is Cohen–Macaulay (see [5] or [6]).

Corollary 3.5. *Suppose R is a local ring, M a finitely generated R-module, $N = R$ and I an ideal such that* $\operatorname{ann}(M) + I$ *is* $\mathfrak{m}$*-primary.*

(a) *If* $\dim(M) < d$, *then for all* i, $\deg(\epsilon^i_I(n)) \leqslant d - 1$.
(b) *If I is an $\mathfrak{m}$-primary ideal and $M = R$, then*

$$ {}^{d}_{I}(n) = \frac{e(I)}{d!} n^d + \textit{lower terms}. $$

Proof. Immediate from Theorem 3.2. $\square$

Corollary 3.6. *Suppose R is a local ring, M a finitely generated R-module, $N = R$ and I an ideal such that* $\operatorname{ann}(M) + I$ *is* $\mathfrak{m}$*-primary. Write* $\delta := \dim(M)$. *If* $\ell(I) \leqslant \delta$, *then* $\deg(\epsilon^i_I(n)) \leqslant \delta - 1$ *for* $i = \delta$ *and* $\deg(\epsilon^\delta_I(n)) = \delta$. *In the latter case, the normalized leading coefficient of* ${}^{\delta}_{I}(n)$ *is* $e(I, M)$.

Proof. The proof follows exactly along the lines of the proof of Theorem 3.2, only, after completing, we take S to be a Gorenstein local ring of dimension δ mapping onto $R/\operatorname{ann}(M)$. $\square$

In the next proposition, we give a condition which guarantees that ${}^{d-1}_{I}(n)$ has degree $d - 1$.

Proposition 3.7. *Let R be a local ring and $I \subseteq R$ an ideal. Assume that M and N are finite R-modules such that I, M, N satisfy our standard support condition. Suppose there exists P in* $\operatorname{Ass}_R(M) \cap \operatorname{Supp}(N)$ *such that* $\dim(R/P) = d - 1$. *Then* $\deg(\epsilon^{d-1}_I(n)) = d - 1$ *and the normalized leading coefficient of* ${}^{d-1}_{I}(n)$ *is at least* $e(I, \operatorname{Hom}(N, M_{d-1}))$.

Proof. Using basic properties of completion, it is not hard to reduce to the case that R is complete. As before, let S be a Gorenstein local ring of dimension d mapping onto R. Now, by Proposition 2.5 and hom-tensor duality, the degree of ${}_I^{d-1}(n)$ is bounded by

$$\max\{\dim(\mathrm{H}_{d-1}(\Gamma_{\mathfrak{m}}(\mathcal{E})^{\vee} \otimes N)), \ell_N(I) - 1\},$$

with equality holding when the maximum occurs with the first term. Since ${}_N(I)$ is bounded above by d, it follows that $\deg(\epsilon_I^{d-1}(n)) = d - 1$, whenever

$$\dim(\mathrm{H}_{d-1}(\Gamma_{\mathfrak{m}}(\mathcal{E})^{\vee} \otimes N)) = d - 1. \tag{3.2}$$

Note that by Proposition 2.5 and Lemma 3.1, the module $\mathrm{H}_{d-1}(\Gamma_{\mathfrak{m}}(\mathcal{E})^{\vee} \otimes N)$ has dimension less than or equal to $d - 1$. On the other hand, local duality implies that

$$\mathrm{H}_{d-i}(\Gamma_{\mathfrak{m}}(\mathcal{E})^{\vee}) = \mathrm{H}_{\mathfrak{m}}^{d-i}(M)^{\vee} = \mathrm{Ext}_S^i(M, S),$$

for $0 \leqslant i \leqslant d$. Now, since P corresponds to a height one prime in S, $\Gamma_{\mathfrak{m}}(\mathcal{E}^{\vee})_P$ is split exact in degrees less than $d - 1$. Using right exactness, it follows from this that

$$(\mathrm{H}_{d-1}(\Gamma_{\mathfrak{m}}(\mathcal{E})^{\vee}) \otimes N)_P = (\mathrm{H}_{d-1}(\Gamma_{\mathfrak{m}}(\mathcal{E})^{\vee}))_P \otimes N_P = \mathrm{Ext}_S^1(M, S)_P \otimes_R N_P.$$

Since $N_P = 0$, if we show that $\mathrm{Ext}_S^1(M, S)_P$ is non-zero, then Eq. (3.2) holds. But this follows since S is Gorenstein. Indeed, if Q is the prime in S corresponding to P, then $\mathrm{height}(Q) = 1$. Thus

$$\mathrm{depth}(S_Q) - \mathrm{depth}(M_Q) = 1 - 0 = 1,$$

so $\mathrm{Ext}_S^1(M, S)_Q = \mathrm{Ext}_S^1(M, S)_P$ does not vanish, which gives what we want. It now follows that ${}_I^{d-1}(n)$ has degree $d - 1$. Moreover, this same calculation shows that a prime Q of dimension $d - 1$ belongs to the support of $\mathrm{H}_{d-1}(\Gamma_{\mathfrak{m}}(\mathcal{E})^{\vee} \otimes_R N)$ if and only if Q belongs to $\mathrm{Ass}(M) \cap \mathrm{Supp}(N)$.

For the statement involving multiplicity, note that by Proposition 2.5, the normalized leading coefficient of ${}_I^{d-1}(n)$ is at least $e(I, \mathrm{H}_{d-1}(\Gamma_{\mathfrak{m}}(\mathcal{E})^{\vee} \otimes N))$. Let P in $\mathrm{Ass}(M) \cap \mathrm{Supp}(N)$ have dimension $d - 1$ and let $Q \subseteq S$ be the corresponding prime. Since S_Q maps onto R_P, $A^{\vee_Q} = A^{\vee_P}$, for any R_P-module A. Thus,

$$\mathrm{Ext}_S^1(M, S)_P = \mathrm{Ext}_S^1(M, S)_Q = \mathrm{H}_Q^0(M_Q)^{\vee_Q} = \mathrm{H}_P^0(M_P)^{\vee_P} = (M_{d-1})_P^{\vee_P}.$$

Finally, note that it follows from the calculation below that $\mathrm{Hom}(N, M_{d-1}) = 0$ and moreover, it follows easily from our standard support condition that $\mathfrak{m}$ is the only prime ideal containing $I + \mathrm{ann}(\mathrm{Hom}(N, M))$. Thus, $I + \mathrm{ann}(\mathrm{Hom}(N, M_{d-1}))$ is $\mathfrak{m}$-primary, so $e(I, \mathrm{Hom}(N, M_{d-1}))$ is defined. Therefore, along similar lines as in the proof of Theorem 3.2, we have

$$\begin{aligned} e(I, \mathrm{H}_{d-1}(\Gamma_{\mathfrak{m}}(\mathcal{E})^{\vee} \otimes_R N)) &= \sum_{\dim P = d-1} \lambda(\mathrm{H}_{d-1}(\Gamma_{\mathfrak{m}}(\mathcal{E})^{\vee})_P \otimes N_P) e(I, R/P) \\ &= \sum_{\dim P = d-1} \lambda(\mathrm{Ext}_S^1(M, S)_P \otimes N_P) e(I, R/P) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\dim P = d-1} \lambda\big((M_{d-1})_P^{\vee_P} \otimes N_P\big) e(I, R/P) \\
&= \sum_{\dim P = d-1} \lambda\big(\operatorname{Hom}(N, M_{d-1})_P\big) e(I, R/P) \\
&= e\big(I, \operatorname{Hom}(N, M_{d-1})\big).
\end{aligned}
$$

Thus, the normalized leading coefficient is at least $e(I, \operatorname{Hom}(N, M_{d-1}))$, which is what we want. $\square$

Corollary 3.8. *Let $I \subseteq R$ be an ideal and M be a finitely generated R-module such that $\operatorname{ann}(M) + I$ is $\mathfrak{m}$-primary. Assume that M has an associated prime P of dimension $d-1$ and take $N = R$. Then, $\deg(\epsilon^{d-1}(n)) = d-1$ and its normalized leading coefficient is at least $e(I, M_{d-1})$. In particular, if $M = R$ and I is an $\mathfrak{m}$-primary ideal, then ${}^{d-1}_{I}(n)$ has degree $d-1$ and its normalized leading coefficient is at least $e(I, R_{d-1})$.*

We now consider the Hilbert function of $\operatorname{Ext}^g(R/I^n, M)$, where M is a module such that the grade of I on M is g. As is well known, these modules are the first non-vanishing extension modules of the form $\operatorname{Ext}^i(R/I^n, M)$. In [5, Theorem 2.4], Kirby showed that the lengths of the modules $\operatorname{Ext}^g(R/I^n, M)$ are ultimately given by a polynomial of degree less than or equal to g. Kirby's proof used his version of Hilbert theory for Artinian modules. In the following theorem, we first give an explicit expression for the lengths of $\operatorname{Ext}^g(R/I^n, M)$, from which it immediately follows that ${}^{g}_{I}(n)$ has degree less than or equal to g. We then determine a necessary and sufficient condition for equality to hold and, in case equality holds, determine the normalized leading coefficient of ${}^{g}_{I}(n)$.

Theorem 3.9. *Let M be a finitely generated R-module and $I \subseteq R$ be an ideal such that $I + \operatorname{ann}(M)$ is $\mathfrak{m}$-primary. Assume $N = R$ and let $\delta := \dim(M)$. Set $g := \operatorname{grade}_I(M)$ and let S be a Gorenstein local ring of dimension δ mapping onto $\hat{R}/\operatorname{ann}(\hat{M})$. Then*:

- (i) *For all large n, ${}^{g}_{I}(n) = \lambda(\operatorname{Ext}_S^{\delta-g}(\hat{M}, S)/I^n \operatorname{Ext}_S^{\delta-g}(\hat{M}, S))$.*
- (ii) *The degree of ${}^{g}_{I}(n)$ is at most g and its normalized leading coefficient is $e(I, \operatorname{Ext}_S^{\delta-g}(\hat{M}, S))$.*
- (iii) *The degree of ${}^{g}_{I}(n)$ equals g if and only if there exists a prime ideal P in $\operatorname{Ass}(M)$ such that $\dim(R/P) = g$.*
- (iv) *If the conditions in* (iii) *hold, the normalized leading coefficient of ${}^{g}_{I}(n)$ is $e(I, M_g)$.*

Proof. Once again, we may complete. Note that since $\operatorname{ann}(M) + I$ is $\mathfrak{m}$-primary, $g := \operatorname{grade}_I(M) = \operatorname{depth}(M)$. Consider $\mathcal{E}$, the minimal injective resolution of M. Then,

$$
\operatorname{Ext}^g\big(R/I^n, M\big) = \mathrm{H}^g\big(\operatorname{Hom}\big(R/I^n, \mathcal{E}\big)\big) = \mathrm{H}^g\big(\operatorname{Hom}\big(R/I^n, \Gamma_{\mathfrak{m}}(\mathcal{E})\big)\big),
$$

the latter equality following from our support condition. Taking Matlis duals, we have

$$
\begin{aligned}
\operatorname{Ext}^g\big(R/I^n, M\big)^\vee &= \mathrm{H}^g\big(\operatorname{Hom}\big(R/I^n, \Gamma_{\mathfrak{m}}(\mathcal{E})\big)\big)^\vee \\
&= \mathrm{H}_g\big(\Gamma_{\mathfrak{m}}(\mathcal{E})^\vee \otimes R/I^n\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{H}_g\big(\big(\cdots \to R^{\mu^{g+1}(\mathfrak{m},M)} \to R^{\mu^{g}(\mathfrak{m},M)} \to 0\big) \otimes R/I^n\big) \\
&= \mathrm{H}^g_{\mathfrak{m}}(M)^{\vee} \otimes R/I^n \\
&= \mathrm{H}^g_{\mathfrak{n}}(M)^{\vee} \otimes R/I^n \\
&= \mathrm{Ext}^{\dim S-g}_S(M,S) \otimes R/I^n.
\end{aligned}
$$

Note, the equalities follow since $I + \mathrm{ann}(M)$ is $\mathfrak{m}$-primary, by exactness of Matlis dual together with adjointness, since $\mu^{g-1}(\mathfrak{m}, M) = 0$, by right exactness of $\otimes$, permanence of local cohomology, and local duality. Thus $^{g}_{I}(n)$ has the required form, so (i) holds. It also follows that $\deg(\epsilon^g_I(n)) = \dim(\mathrm{Ext}^{\delta-g}_S(M,S))$ and that the normalized leading coefficient of $^{g}_{I}(n)$ equals $e(IS, \mathrm{Ext}^{\delta-g}_S(M,S))$. In particular, the second part of (ii) holds.

For the first part of (ii) regarding the degree of $^{g}_{I}(n)$, i.e., $\dim(\mathrm{Ext}^{\delta-g}_S(M,S))$, we are free to work with primes in S. If Q is a prime of S of dimension greater than g (and hence height less than $\delta - g$), then $\mathrm{Ext}^{\delta-g}_S(M,S)_Q = 0$, since S_Q has injective dimension less than $\delta - g$. This shows that $\mathrm{Ext}^{\delta-g}_S(M,S)$ has dimension less than or equal to g, and thus gives $\deg(\epsilon^g_I(n)) \leqslant g$, so the first statement in (ii) holds.

Concerning the possibility of equality holding, let $Q \subseteq S$ be a prime ideal with dimension g. Then $\mathrm{height}(Q) = \delta - g$, so $\mathrm{depth}(S_Q) = \delta - g$. Thus, since S is Gorenstein, $\mathrm{Ext}^{\delta-g}_S(M,S)_Q = 0$ if and only if $\mathrm{depth}(M_Q) = 0$, i.e., if and only if $Q \in \mathrm{Ass}(M)$. Thus, $\deg(\epsilon^g_I(n)) = g$ if and only if there exists a prime Q of dimension g belonging to $\mathrm{Ass}(M)$, so (iii) holds.

Finally, for part (iv), suppose $\deg(\epsilon^g_I(n)) = g$. To calculate the normalized leading coefficient of $^{g}_{I}(n)$, we proceed as before via the associativity formula to get

$$
\begin{aligned}
\big(I, \mathrm{Ext}^{\delta-g}(M,S)\big) &= \sum_{\dim(R/P)=g} \lambda\big(\mathrm{Ext}^{\delta-g}(M,S)_P\big)e(I, R/P) \\
&= \sum_{\dim(R/P)=g} \lambda\big(H^0_P(M)_P\big)e(I, R/P) \\
&= \sum_{\dim(R/P)=g} \lambda\big((M_g)_P\big)e(I, R/P) \\
&= e(I, M_g),
\end{aligned}
$$

and the proof is complete. □

Corollary 3.10. *Assume that* $N = R$ *and* $\mathrm{ann}(M) + I$ *is* $\mathfrak{m}$*-primary. If* M *is Cohen–Macaulay and* $\dim(M) = \delta$*, then* $\deg(\epsilon^\delta(n)) = \delta$ *and the normalized leading coefficient of* $^{\delta}(n)$ *is* $e(I, M)$.

Proof. Since M is Cohen–Macaulay, $\delta = \mathrm{depth}(M)$, so it follows immediately from the previous theorem that $\deg(\epsilon^\delta(n)) = \delta$. Moreover, since M is Cohen–Macaulay, M is unmixed, so $M_\delta = M$. Thus, $e(I, M)$ is the normalized leading coefficient of $^{\delta}_{I}(n)$. □

In our next result, we use Matlis duality to give a version for $^{i}_{I}(n)$ of Theorems 3.3 and 3.4 from [4]. Note that $\overline{J}$ denotes the integral closure of an ideal J.

Theorem 3.11. *Assume that R is analytically irreducible. Let I be an ideal having analytic spread d, let M be a finite R-module such that $\operatorname{ann}(M) + I$ is $\mathfrak{m}$-primary, and take $N = R$. Assume further that one of the following conditions hold.*

(i) $I = \mathfrak{m}K$ *for some ideal* $K \subseteq R$.
(ii) $(\mathfrak{m}I^n : \mathfrak{m}) = I^n$, *large* n.
(iii) $(I^n : \mathfrak{m}) \subseteq \overline{I^n}$ *for some* n *and* R *is quasi-unmixed.*

Then, for $\operatorname{grade}_I(M) < i \leqslant i.d.(M)$, $i = d$, $\deg(\epsilon_I^i(n)) = d - 1$.

Proof. We may complete and assume $R = \hat{R}$. Also note that by our support hypothesis, $\operatorname{grade}_I(M) = \operatorname{depth}(M)$, so that $\operatorname{Ext}^i(R/I^n, M) = 0$, for $i < \operatorname{depth}(M)$.

Now assume $\operatorname{depth}(M) = i = d$. Consider a minimal injective resolution of M

$$0 \to M \to Q_0 \xrightarrow{\delta^0} Q_1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{i-2}} Q_{i-1} \xrightarrow{\delta^{i-1}} Q_i \xrightarrow{\delta^i} Q_{i+1} \to \cdots. \tag{3.3}$$

By dimension shifting, it follows that $\operatorname{Ext}^i(R/I^n, M) = \operatorname{Ext}^1(R/I^n, C)$, where C is the $(i-1)$st cosyzygy of M. Let $L := \mathrm{H}^0_{\mathfrak{m}}(C)$ be the largest Artinian submodule of C. Then $L = 0$, since $i \geqslant \operatorname{depth}(M)$ (and since by [3, Theorem 1.1], once a prime gives rise to a non-zero Bass number at some stage in the minimal injective resolution of M, it has non-zero Bass number at all further non-zero stages in the resolution). We have the exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}\bigl(R/I^n, L\bigr) \to \operatorname{Hom}\bigl(R/I^n, C\bigr) \to \operatorname{Hom}\bigl(R/I^n, C/L\bigr) \\ &\to \operatorname{Ext}^1\bigl(R/I^n, L\bigr) \to \operatorname{Ext}^1\bigl(R/I^n, C\bigr). \end{aligned}$$

Now, by Proposition 2.5 and Corollary 3.4 above, the degree of ${}^{i}_{I}(n)$, which is the degree of the polynomial giving the lengths of $\operatorname{Ext}^1(R/I^n, C)$, is less than or equal to $d - 1$. We now note that our support hypothesis implies that $\operatorname{Hom}(R/I^n, C/L) = 0$. Indeed, suppose $c \in C$ is such that $I^n \cdot c \subseteq L$. For any $x \in \operatorname{ann}(M)$, $M_x = 0$, so by minimality of (3.3), $C_x = 0$. Thus, there exists a $q > 0$ such that $(\operatorname{ann}(M))^q \cdot c = 0$. Since $I + \operatorname{ann}(M)$ is $\mathfrak{m}$-primary, it follows that for p sufficiently large, $\mathfrak{m}^p \cdot c \subseteq L$, from which it follows that $c \in L$. Thus, $\operatorname{Hom}(R/I^n, C/L) = 0$, as claimed. It follows that if we show that the degree of the polynomial giving the lengths of $\operatorname{Ext}^1(R/I^n, L)$ is $d - 1$, then $\deg(\epsilon_I^i(n)) = d - 1$, which is what we want. But the lengths of the $\operatorname{Ext}^1(R/I^n, L)$ are the same as the lengths of their Matlis duals which are $\operatorname{Tor}_1(R/I^n, L^\vee)$, where $L^\vee$ is a finitely generated R-module. Since R is a domain, $L^\vee$ clearly has a rank. The result now follows from Theorems 3.3 and 3.4 in [4]. □

Remark 3.12. The main point about Theorem 3.11 is the following. By Corollary 3.4, we know that for i in the indicated range, $\deg(\epsilon_I^i(n)) \leqslant d - 1$. The conditions (i)–(iii) stated in Theorem 3.11 guarantee that $\deg(\epsilon_I^i(n))$ does not drop below $d - 1$.

4. Iterated applications

In this section we consider functions giving lengths of iterated expressions of the form $\operatorname{Ext}^j(\operatorname{Ext}^i(N/I^nN, M), M\ )$, for finitely generated R-modules N, M, and $M\ $ and $I \subseteq R$ an ideal such that I, M, N satisfy our standard support condition. Note that when R is Gorenstein and $M\ = M = N = R$, then one has that R/I^n is isomorphic to $\mathrm{Ext}^d(\mathrm{Ext}^d(R/I^n, R), R)$, so the two length functions are actually the same. Using the results from [6], it is not hard to show that, in the presence of our usual support condition, the lengths of $\mathrm{Ext}^j(\mathrm{Ext}^i(N/I^nN, M), M\ )$ are given by a polynomial in n, for n large. Our work below will characterize when this polynomial has degree d and show that its normalized leading coefficient can be expressed in terms of the multiplicity of the ideal on an iterated Ext module derived from N, M and $M\ $. In particular, we obtain as a corollary that for *any* local ring R and any $\mathfrak{m}$-primary ideal $I \subseteq R$, the degree and normalized leading coefficients for the Hilbert polynomials giving the lengths $\lambda(R/I^n)$ and $\lambda(\mathrm{Ext}^d(\mathrm{Ext}^d(R/I^n, R), R))$ remain the same.

Remark 4.1. We start with a lemma that is similar in spirit to Lemma 2.3. We set some notation for the lemma. Suppose, just as in Lemma 2.3, $I \subseteq R$ is an ideal and U, V, W are submodules of a common finitely generated R-module so that $W \subseteq V$. For $n > 0$, set $L_n := (U + I^nV)/I^nW$. Let $\mathcal{C}$ be a co-chain complex of finitely generated free R-modules and assume that the lengths of the cohomology modules $\mathrm{H}^j(L_n \otimes \mathcal{C})$ are finite for $j > 0$. Then by [6, Proposition 3(b)] the lengths of these cohomology modules in are given by a rational polynomial for n large. We write $Q_j(n)$ for this polynomial.

Lemma 4.2. *Let $(R, \mathfrak{m})$ be a local ring of dimension d and $I \subseteq R$ an ideal. Let $L_n, \mathcal{C}$, and $Q_j(n)$ be as in Remark* 4.1. *The following statements are equivalent*:

(a) $\deg(Q_j(n)) = d$.
(b) $\mathrm{H}^j(\mathcal{C} \otimes U)$ *has dimension d.*

Proof. We start by tensoring the short exact sequence

$$0 \to \frac{I^nV}{I^nW} \to L_n \to \frac{U + I^nV}{I^nV} \to 0,$$

with $\mathcal{C}$ to get the long exact sequence

$$\cdots \to \mathrm{H}^j\ \ \frac{I^nV}{I^nW} \otimes \mathcal{C}\ \ \to \mathrm{H}^j(L_n \otimes \mathcal{C})$$
$$\to \mathrm{H}^j\ \ \frac{U + I^nV}{I^nV} \otimes \mathcal{C}\ \ \to \mathrm{H}^{j+1}\ \ \frac{I^nV}{I^nW} \otimes \mathcal{C}\ \ \to \cdots$$

(cf. [6, Proposition 3]).

We claim that in order to determine the coefficient of degree d of $Q_j(n)$, and in particular, to determine if that coefficient is non-zero, we only need to consider the same for $\mathrm{H}^j(\{(U + I^nV)/I^nV\} \otimes \mathcal{C})$. Indeed, the degree of the Hilbert polynomial which gives the length of $\mathrm{H}^j((I^nV/I^nW) \otimes \mathcal{C})$ is *less* than d, for all j since

$$\bigoplus_{n \geqslant 0} \mathrm{H}^j\big((I^nV/I^nW) \otimes \mathcal{C}\big) = \mathrm{H}^j\ \ \bigoplus_{n \geqslant 0}\big(I^nV/I^nW\big) \otimes \mathcal{C}$$

is a finite graded module over the Rees ring of I.

In order to determine when the length of $\mathrm{H}^j(((U+I^nV)/I^nV)\otimes\mathcal{C})$ is eventually given by a polynomial of degree d (and to determine its leading coefficient) we will ultimately appeal to [6, Proposition 3(c)]. More precisely, we claim that in order to see that the degree of the polynomial in question is d, it is enough to check that $\mathrm{H}^j(\mathcal{C}\otimes U)$ has dimension d. Indeed, following the spirit of Lemma 2.3, we have the isomorphisms

$$\frac{U+I^nV}{I^nV}\cong\frac{U}{U\cap I^nV}\cong\frac{U}{I^{n-t}(U\cap I^tV)},$$

for t large enough and $n\geqslant t$. This leads to the short exact sequence

$$0\to\frac{U\cap I^tV}{I^{n-t}(U\cap I^tV)}\to\frac{U}{I^{n-t}(U\cap I^tV)}\to\frac{U}{U\cap I^tV}\to 0.$$

Tensoring this sequence with $\mathcal{C}$ and using the resulting long exact sequence in homology, we see that the polynomials giving the lengths of the modules

$$\mathrm{H}^j\big(\mathcal{C}\otimes\big\{\big(U+I^nV\big)/I^nV\big\}\big)\quad\text{and}\quad\mathrm{H}^j\big(\mathcal{C}\otimes\big\{\big(U\cap I^tV\big)/I^{n-t}\big(U\cap I^tV\big)\big\}\big)$$

simultaneously have degree d, because $U/(U\cap I^tV)$ has length independent of n. Furthermore, since $U_P=(U\cap I^tV)_P$ for all primes $P=\mathfrak{m}$, it follows that $\mathrm{H}^j(\mathcal{C}\otimes U)$ has dimension d if and only if $\mathrm{H}^j(\mathcal{C}\otimes(U\cap I^tV))$ has dimension d. But, the polynomial giving the lengths of

$$\mathrm{H}^j\big(\big\{\big(U\cap I^tV\big)/I^{n-t}\big(U\cap I^tV\big)\big\}\otimes\mathcal{C}\big)$$

has degree equal to d if and only if $\mathrm{H}^j(\mathcal{C}\otimes(U\cap I^tV))$ has dimension d, by [6, Proposition 3(c)]. Thus the polynomial $Q_j(n)$ has degree d if and only if $\mathrm{H}^j(\mathcal{C}\otimes U)$ has dimension d. □

Theorem 4.3. *Let $(R,\mathfrak{m})$ be a local ring of dimension d and $I\subseteq R$ an ideal. Let N, M, and M be finitely generated R-modules such that I, N, M satisfy our standard support condition. Fix $i, j\geqslant 0$.*

(i) *The function $\lambda(\mathrm{Ext}^j(\mathrm{Ext}^i(N/I^nN,M),M))$ is given by a rational polynomial $Q(n)$, for n large.*
(ii) *The following are equivalent*:
  (a) $\deg(Q(n))=d$.
  (b) $i,j\geqslant d$ *and* $\dim(\mathrm{Ext}^{j-d}(\mathrm{Ext}^{i-d}(N,M),M))=d$.
(iii) *If* (a) *and* (b) *hold in* (ii)*, then the normalized leading coefficient of $Q(n)$ equals $e(I,\mathrm{Ext}^{j-d}(\mathrm{Ext}^{i-d}(N,M),M))$.*

Proof. We may assume that R is complete. As before, let S be a Gorenstein local ring of dimension d mapping onto R. Now, let $\mathcal{E}$ and $\mathcal{E}$ respectively denote the minimal injective resolutions of M and M. Set $\mathcal{D}:=\Gamma_{\mathfrak{m}}(\mathcal{E})^\vee$. Thus, just as in the proof of Lemma 3.1, for any prime P of dimension d, $\mathcal{D}_P$ is split exact in degrees less than d and if we truncate $\mathcal{D}_P$ at the dth spot, we get an R_P-free resolution of $\mathrm{Hom}_S(M,S)_P$. Set $\mathcal{C}:=\mathrm{Hom}(\mathcal{D},R)$. Therefore,

$$\mathcal{C}:\quad\cdots\to R^{\mu_{j-1}(\mathfrak{m},M)}\to R^{\mu_j(\mathfrak{m},M)}\to R^{\mu_{j+1}(\mathfrak{m},M)}\to\cdots.$$

Now, we have

$$\begin{aligned}\operatorname{Ext}^j\bigl(\operatorname{Ext}^i\bigl(N/I^nN, M\bigr), M\ \bigr) &= \mathrm{H}^j\bigl(\operatorname{Hom}\bigl(\operatorname{Ext}^i\bigl(N/I^nN, M\bigr), \mathcal{E}\ \bigr)\bigr)\\ &= \mathrm{H}^j\bigl(\operatorname{Hom}\bigl(\operatorname{Ext}^i\bigl(N/I^nN, M\bigr), \Gamma_{\mathfrak{m}}(\mathcal{E}\ )\bigr)\bigr)\\ &= \mathrm{H}^j\bigl(\operatorname{Ext}^i\bigl(N/I^nN, M\bigr)^\vee \otimes \mathcal{C}\bigr).\end{aligned}$$

As noted in Remark 2.2, it follows from [6] that, for some n_0,

$$\operatorname{Ext}^i\bigl(N/I^{n+n_0}N, M\bigr)^\vee = \frac{U + I^nV}{I^nW} \quad \text{for } n \qquad 0,$$

where $U = \mathrm{H}_i(\Gamma_{\mathfrak{m}}(\mathcal{E})^\vee \otimes N)$. By [6, Proposition 3(b)], the homology modules in the complex $\mathcal{C} \otimes \{(U + I^nV)/I^nW\}$ have polynomial growth. Thus, $Q(n)$ exists, which gives (i).

To prove (ii), we proceed in three steps. For the first step, we show that if $i < d$, then $\deg(Q(n)) < d$. To see this, first note that by Theorem 3.2, the polynomial giving the lengths of $\operatorname{Ext}^i(N/I^nN, M)^\vee$ has degree less than d, if $i < d$. Since the homology modules in $\operatorname{Ext}^i(N/I^nN, M)^\vee \otimes \mathcal{C}$ are subquotients of finite sums of $\operatorname{Ext}^i(N/I^nN, M)^\vee$, it follows that $\deg(Q(n)) < d$ for $i < d$, which is what we want. We now assume $i \geqslant d$.

For our second step, we show that if $j < d$, then $\deg(Q(n)) < d$. By Lemma 4.2, we must show that $\mathrm{H}^j(\mathcal{C} \otimes U)$ has dimension less than d. Let P be a prime of dimension d. Using exactness of the localization and the Matlis duality functor $\vee_P$ locally in codimension zero,

$$\lambda\bigl(\mathrm{H}^j(\mathcal{C} \otimes U)_P\bigr) = \lambda\bigl(\mathrm{H}^j(\mathcal{C}_P \otimes_{R_P} U_P)\bigr) = \lambda\bigl(\mathrm{H}_j(\mathcal{C}_P \otimes_{R_P} U_P)^{\vee_P}\bigr). \tag{4.1}$$

Since $\mathcal{C} = \operatorname{Hom}(\mathcal{D}, R)$, $\operatorname{Hom}(\mathcal{C}, L) = \mathcal{D} \otimes L$, for any R-module L. Thus, using Hom-tensor duality over R_P, we have $(\mathcal{C}_P \otimes_{R_P} U_P)^{\vee_P} = \mathcal{D}_P \otimes_{R_P} {U_P}^{\vee_P}$. Since $\mathcal{D}_P$ is split exact in degrees less than d, it follows that $\dim(\mathrm{H}^j(\mathcal{C} \otimes U)) < d$, for $j < d$. Thus, we now have that $\deg(Q(n)) < d$, if $j < d$, which is what we want.

For our final step in the proof of (ii), we assume $j \geqslant d$, $i \geqslant d$ and prove that $\deg(Q(n)) = d$ if and only if $\dim(\operatorname{Ext}^{j-d}(\operatorname{Ext}^{i-d}(N, M), M\ )) = d$. Using what we have just observed about the relation between $\mathcal{C}$ and $\mathcal{D}$, we may extend Eq. (4.1) to get for any P of dimension d

$$\lambda\bigl(\mathrm{H}^j(\mathcal{C} \otimes U)_P\bigr) = \lambda\bigl(\mathrm{H}_j\bigl(\mathcal{D}_P \otimes_{R_P} {U_P}^{\vee_P}\bigr)\bigr). \tag{4.2}$$

Recall from the proof of Lemma 3.1 that ${U_P}^{\vee_P} = (\operatorname{Ext}_R^{i-d}(N, M))_P$ and

$$\mathrm{H}_j\bigl(\mathcal{D}_P \otimes_{R_P} {U_P}^{\vee_P}\bigr) = \operatorname{Tor}_{j-d}^{R_P}\bigl((M_P)^{\vee_P}, {U_P}^{\vee_P}\bigr), \quad \text{for } j \geqslant d.$$

Thus, extending Eq. (4.2), we have

$$\begin{aligned}\lambda\bigl(\mathrm{H}^j(\mathcal{C} \otimes U)_P\bigr) &= \lambda\bigl(\operatorname{Tor}_{j-d}^{R_P}\bigl((M_P)^{\vee_P}, {U_P}^{\vee_P}\bigr)\bigr) && (4.3)\\ &= \lambda\bigl(\operatorname{Ext}_{R_P}^{j-d}\bigl(U_P^{\vee_P}, M_P\bigr)\bigr) && (4.4)\\ &= \lambda\bigl(\operatorname{Ext}_{R_P}^{j-d}\bigl(\operatorname{Ext}_{R_P}^{i-d}(N_P, M_P), M_P\bigr)\bigr) && (4.5)\\ &= \lambda\bigl(\operatorname{Ext}_R^{j-d}\bigl(\operatorname{Ext}_R^{i-d}(N, M), M\ \bigr)_P\bigr). && (4.6)\end{aligned}$$

Here we have used the invariance of length under Matlis duality as well as the duality between Tor and Ext. We now have that a prime ideal of maximal dimension belongs to the support of $\mathrm{H}^j(\mathcal{C}\otimes U)$ if and only if it belongs to the support of $\mathrm{Ext}^{j-d}(\mathrm{Ext}^{i-d}(N,M),M)$. Therefore, $\dim(\mathrm{H}^j(\mathcal{C}\otimes U))=d$ if and only if $\dim(\mathrm{Ext}^{j-d}(\mathrm{Ext}^{i-d}(N,M),M))=d$. Thus, $\deg(Q(n))=d$ if and only if $\dim(\mathrm{Ext}^{j-d}(\mathrm{Ext}^{i-d}(N,M),M))=d$, which is what we wanted to show. Part (ii) of the theorem now follows immediately by combining the three steps.

Finally for (iii), assume that $\deg(Q(n))=d$. From Lemma 4.2 and its proof we have $\dim(\mathrm{H}^j(\mathcal{C}\otimes U))=\dim(\mathrm{H}^j(\mathcal{C}\otimes(U\cap I^tV)))=d$. In fact, as noted in the proof of Lemma 4.2, $Q((n))$ and the polynomial giving the lengths of the modules

$$\mathrm{H}^j\big(\mathcal{C}\otimes\big\{\big(U\cap I^tV\big)/I^{n-t}\big(U\cap I^tV\big)\big\}\big)$$

differ by a polynomial of degree less than d. By [6, Proposition 3(c)],

$$\mathrm{H}^j\big(\mathcal{C}\otimes\big\{\big(U\cap I^tV\big)/I^{n-t}\big(U\cap I^tV\big)\big\}\big)$$

has the form $(A+I^nB)/I^nC$, with $C\subseteq B$ and $A=\mathrm{H}^j(\mathcal{C}\otimes(U\cap I^tV))$. Thus, by Lemma 2.3, the normalized leading coefficient of $Q(n)$ is $e(I;\mathrm{H}^j(\mathcal{C}\otimes(U\cap I^tV)))$. Using Eq. (4.6) above in the associativity formula gives

$$\begin{aligned}
e\big(I;\mathrm{H}^j\big(\mathcal{C}\otimes\big(U\cap I^tV\big)\big)\big)&=\sum_{\dim P=d}e(I;R/P)\lambda\big(\mathrm{H}^j\big(\mathcal{C}\otimes\big(U\cap I^tV\big)\big)_P\big)\\
&=\sum_{\dim P=d}e(I;R/P)\lambda\big(\mathrm{H}^j(\mathcal{C}\otimes U)_P\big)\\
&=\sum_{\dim P=d}e(I;R/P)\lambda\big(\mathrm{Ext}_R^{j-d}\big(\mathrm{Ext}_R^{i-d}(N,M),M\big)_P\big)\\
&=e\big(I;\mathrm{Ext}^{j-d}\big(\mathrm{Ext}^{i-d}(N,M),M\big)\big),
\end{aligned}$$

which completes the proof. □

Remark 4.4. Recall that a module C is said to be *semi-dualizing* if the natural map from R to $\mathrm{Hom}_R(C,C)$ is an isomorphism and $\mathrm{Ext}^i(C,C)=0$, for all $i>0$. For more information on semi-dualizing modules, see [2], where examples are given of semi-dualizing modules that are not dualizing modules (see, [2, p. 1874]).

The more general first part of the following corollary answers a question posed to the second author by S. Sather-Wagstaff, while the second part of the following corollary generalizes what is obvious in the case that R is Gorenstein. Both parts follow immediately from Theorem 4.3.

Corollary 4.5. *Let $(R,\mathfrak{m},k)$ be a local ring of dimension d. Let C be a semi-dualizing module and $I\subseteq R$ an $\mathfrak{m}$-primary ideal. Then for all large n, we have*

$$\lambda\big(\mathrm{Ext}^d\big(\mathrm{Ext}^d\big(R/I^n,C\big),C\big)\big)=\frac{e(I)}{d!}\cdot n^d+\textit{lower degree terms}.$$

In particular, the Hilbert polynomials for R/I^n and $\mathrm{Ext}^d(\mathrm{Ext}^d(R/I^n,R),R)$ have the same degree and same normalized leading coefficient.

We close this section by giving a version of Theorem 3.9 for an iterated Ext in degree g, where $g := \mathrm{grade}_I(M)$. Denote $E^2(-) := E \circ E$, where the functor $E := \mathrm{Ext}^g(-, M)$. We will use the following notation in the proposition below. Let S be a Gorenstein local ring of dimension δ mapping onto $\hat{R}/\operatorname{ann}(\hat{M})$ and set $C := \mathrm{Ext}_S^{\delta-g}(\hat{M}, S)$, where $\delta := \dim(M)$. Set $T_2(C) := C \otimes C$.

Proposition 4.6. *Let M be a finite R-module of dimension δ, depth g and let I be an ideal such that $I + \operatorname{ann} M$ is $\mathfrak{m}$-primary. Then with the notation introduced in the paragraph above,*

- (i) $E^2(R/I^n) \cong \mathrm{Hom}(C, E(R/I^n)^\vee) \cong \mathrm{Hom}(C, C/I^nC)$.
- (ii) *The polynomial $Q_2(n)$ which agrees with $\lambda(E^2(R/I^n))$ for n large has degree* $\dim(C)$ *and normalized leading coefficient $e(I, \mathrm{Hom}(C, C))$.*
- (iii) $\deg(Q_2(n)) \leqslant g$ *and equality holds if and only if* $\mathrm{Ass}(M)$ *contains a prime of dimension g.*
- (iv) *If* $\mathrm{Ass}(M)$ *contains a prime of dimension g, the normalized leading coefficient of $Q_2(n)$ is $e(I, \mathrm{Hom}(M_g, M_g))$.*

Proof. Again, we may assume R is complete. To prove (i), we begin by noting that it follows from the proof of Theorem 3.9 that $E(R/I^n)^\vee$ is isomorphic to $C \otimes R/I^n$. Thus the second expression in (i) for $E^2(R/I^n)$ follows immediately from the first. The first expression for $E^2(R/I^n)$ follows along the lines of the proof of Theorem 3.9. Following the same argument as in the proof of Theorem 3.9, with R/I^n replaced by $E(R/I^n)$, we get that

$$E^2\big(R/I^n\big)^\vee = \mathrm{Ext}_S^{\delta-g}(M, S) \otimes E\big(R/I^n\big) = C \otimes E\big(R/I^n\big),$$

which, by Matlis duality, gives (i).

For (ii), by direct computation from a finite presentation of C we get

$$\mathrm{Hom}\big(C, C/I^nC\big) = \big(U + I^{n-n_0}V\big)/I^{n-n_0}W,$$

for some n_0 and $n \geqslant n_0$, where $U := \mathrm{Hom}(C, C)$ and $W \subseteq V$ are two finite R-modules contained in a direct sum of finitely many copies of C. Thus,

$$\dim(V) \leqslant \dim(C) = \dim(U)$$

and it follows from this that ${}_V(I) \leqslant \dim(U)$. By Lemma 2.3, the degree of $Q_2(n)$ is $\dim(U) = \dim(C)$ and its normalized leading coefficient is

$$e(I, U) = e\big(I, \mathrm{Hom}(C, C)\big),$$

so (ii) holds.

For (iii) we note that in the proof of Theorem 3.9 it is shown that $\dim(C) \leqslant g$ and equality holds if and only if M has an associated prime of dimension g. Thus (iii) follows from (ii).

Finally, to see (iv), suppose $\mathrm{Ass}(M)$ contains a prime of dimension g, i.e., $\dim(C) = g$. As in the proof of Theorem 3.9, for any prime P of dimension g, we have that $C_P = (M_g)_P$, so by the associativity formula

$$e\big(I, \mathrm{Hom}(C, C)\big) = e\big(I, \mathrm{Hom}(M_g, M_g)\big)$$

and the proof is now complete. $\square$

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