Castelnuovo Regularity and Graded Rings Associated to an Ideal

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ABSTRACT. We compare the Castelnuovo regularity defined with respect to different homogeneous ideals in a graded ring and use the result we obtain to prove a generalized Goto-Shimoda theorem for ideals of positive height in a Cohen-Macaulay local ring.

1. INTRODUCTION

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and $I \subseteq R$ an ideal. A number of papers in the past ten years or so have studied the transfer of the Cohen-Macaulay property of R to various graded rings associated to I, with particular attention being paid to $\mathscr{G} = \mathscr{G}(I)$ and $\mathscr{R} = \mathscr{R}(I)$ — the associated graded ring and the Rees ring of R with respect to I. In [H], Huneke showed that $\mathscr{G}$ is Cohen-Macaulay whenever $\mathscr{R}$ is Cohen-Macaulay (and the ideal has positive height) and pointed out that the converse need not hold. Since then, numerous authors have studied additional conditions required for $\mathscr{R}$ to be Cohen-Macaulay when $\mathscr{G}$ is Cohen-Macaulay. One of the most important theorems to emerge from these endeavors is the so-called Goto-Shimoda theorem ([GS, Theorem 3.1]) which we now state.

Theorem (Goto-Shimoda). *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with infinite residue field and $I \subseteq R$ an $\mathfrak{m}$-primary ideal. Then $\mathscr{R}$ is Cohen-Macaulay if and only if $\mathscr{G}$ is Cohen-Macaulay and $J I^{d-1} = I^d$ for every minimal reduction J of I.*

It is fair to say that [GS] has provided the impetus for a large amount of research. Notable among subsequent endeavors is [GHO], where the Goto-Shimoda theorem was extended to equimultiple ideals (i.e., ideals whose height equals their analytic spread). The theorem in [GHO] reads exactly the same as the one above, only the assumption that I is $\mathfrak{m}$-primary is replaced by the (more general) assumption that I is equimultiple and $d = \dim(R)$ is replaced by $s = s(I)$, the analytic spread of I. Little progress was made on extending the

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original result further, until the recent paper [GH]. There, using the concept of *analytic deviation* introduced by Huckaba and Huneke in [HH1] and [HH2], the authors extended the Goto-Shimoda theorem to ideals with analytic deviation one (i.e., height$(I) + 1 = s(I)$). Goto and Huckaba's theorem reads as follows:

Theorem (Goto-Huckaba). *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field and $I \subseteq R$ an ideal having positive height which is generically a complete intersection and has analytic deviation one. Let $s = s(I)$. Then $\mathscr{R}$ is Cohen-Macaulay if and only if $\mathscr{G}$ is Cohen-Macaulay and $JI^{s-1} = I^s$ for every minimal reduction J of I.*

In the meantime, a different approach to the question of when $\mathscr{R}$ is Cohen-Macaulay was taken by Trung and Ikeda in [TI]. Using graded local cohomology of $\mathscr{G}$ with respect to its homogeneous maximal ideal, they were able to characterize completely when $\mathscr{R}$ is Cohen-Macaulay for any ideal in any local ring. A particular case of their theorem, relevant to the discussion at hand, is the result below. In the statement of this result, we write $a_d(\mathfrak{M}, \mathscr{G})$ for the smallest integer k such that $H^d_{\mathfrak{M}}(\mathscr{G})_j = 0$, for all $j > k$, where $H^d_{\mathfrak{M}}(\mathscr{G})$ denotes the d th local cohomology module of $\mathscr{G}$ with respect to its homogeneous maximal ideal $\mathfrak{M}$.

Theorem (Trung-Ikeda). *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring and $I \subseteq R$ an ideal having positive height. Then $\mathscr{R}$ is Cohen-Macaulay if and only if $\mathscr{G}$ is Cohen-Macaulay and $a_d(\mathfrak{M}, \mathscr{G}) < 0$.*

This result, aside from its intrinsic elegance, has been extremely useful. In fact, using it, one may readily recover the original Goto-Shimoda theorem and the version given in [GHO]. Moreover, the theorem of Trung-Ikeda also plays a crucial role in the main result of [GH]. At first blush, there may not seem to be a connection between the Goto-Shimoda theorem and the Trung-Ikeda theorem, but indeed there is. This connection is provided by the following result [T, Proposition 3.2].

Proposition (Trung). *Let $(R, \mathfrak{m})$ be a local ring with infinite residue field and $I \subseteq R$ an ideal. For any minimal reduction $J \subseteq I$, write $r_J(I)$ for the least integer n such that $JI^n = I^{n+1}$ and let $s = s(I)$. Then*

$$a_s(\mathscr{G}_+, \mathscr{G}) + s \leq r_J(I) \leq \max\{a_i(\mathscr{G}_+, \mathscr{G}) + i \mid i = 0, \ldots, s\}.$$

In this proposition, $a_i(\mathscr{G}_+, \mathscr{G})$ is the least integer k_i such that $H^i_{\mathscr{G}_+}(\mathscr{G})_j = 0$ for all $j > k_i$, where $H^i_{\mathscr{G}_+}(\mathscr{G})$ denotes the i th local cohomology module of $\mathscr{G}$ with respect to $\mathscr{G}_+$. Using the proposition and the theorem of Trung-Ikeda, the Goto-Shimoda result is immediate and, using some fairly standard reduction techniques, the equimultiple case can be reduced to the $\mathfrak{m}$-primary case and so may also be recovered. (The interested reader is urged to read the proof in [GHO], where an extremely nice noncohomological argument is given.) The problem of deducing a general Goto-Shimoda theorem should now be apparent. The Trung-Ikeda theorem provides a means to deduce that $\mathscr{R}$ is Cohen-Macaulay based upon information concerning the local cohomology modules of $\mathscr{G}$ with respect to $\mathfrak{M}$, while in turn, the reduction number of I is determined by information based upon the local cohomology modules of $\mathscr{G}$ with respect to $\mathscr{G}_+$. It is the goal of the present paper to establish a connection

between these modules, in as much as we seek to relate the vanishing of the degree j components of the module $H^d_{\mathfrak{M}}(\mathscr{G})$ with the vanishing of the degree j components of the modules $H^i_{\mathscr{G}_+}(\mathscr{G})$. This is done in Proposition 2.1 below. Using this result we are able to prove the following generalization of the Goto-Shimoda theorem. In the statement of the-result, we let $\mathscr{A}(I)$ denote the necessarily finite set of primes P containing I for which $s(I_P) = \dim(R_P)$.

Theorem 2.3 (Generalized Goto-Shimoda). *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field and $I \subseteq R$ an ideal having positive height. Then the following statements are equivalent:*

(i) *$\mathscr{R}$ is Cohen-Macaulay.*

(ii) *$\mathscr{G}$ is Cohen-Macaulay and $\mathrm{rd}_{J_P}(I_P) \leq s(I_P) - 1$, for all prime ideals P containing I and all minimal reductions J_P of I_P.*

(iii) *$\mathscr{G}$ is Cohen-Macaulay and $\mathrm{rd}(I_P) \leq s(I_P) - 1$, for all prime ideals $P \in \mathscr{A}(I)$.*

Here we are using $\mathrm{rd}(I)$ to denote the *reduction number of I*, which is defined to be the least integer $\mathrm{rd}_J(I)$ such that $J \subseteq I$ is a minimal reduction. Note that if we let $s = s(I)$ and assume that I is equimultiple or generically a complete intersection with analytic deviation one, then the condition $\mathrm{rd}(I_P) \leq s(I_P) - 1$ holds for all prime ideals P containing I if we assume $JI^{s-1} = I^s$ for every minimal reduction $J \subseteq I$, so we have recovered the previously mentioned extensions of the original Goto-Shimoda theorem.

We now briefly describe the notation and terminology used in this paper. Let $(R, \mathfrak{m})$ be a d-dimensional local ring with residue field k. For an ideal $I \subseteq R$, we consider $\mathscr{G} := \mathscr{G}(I) := \bigoplus_{j \geq 0} I^j/I^{j+1}$, the *associated graded ring* of R with respect to I and $\mathscr{R} := \mathscr{R}(I) := \bigoplus_{j \geq 0} I^j$, the *Rees ring* of R with respect to I. Closely related is the *fiber ring* of R with respect to I, $\mathscr{F} := \mathscr{F}(I) := \bigoplus_{j \geq 0} I^j/\mathfrak{m}I^j$. Recall that $s(I)$, the *analytic spread of I*, is the Krull dimension of $\mathscr{F}$. Writing $\mathscr{A}(I)$ for the prime ideals containing I for which $s(I_P) = \dim(R_P)$, it follows from [Mc, Chapter 4] that if R is *quasi-unmixed* (hence, in particular, if R is Cohen-Macaulay), then $P \in \mathscr{A}(I)$ if and only if there exists a minimal prime $\mathfrak{P}$ of $I\mathscr{R}$ such that $\mathfrak{P} \cap R = P$. Therefore, $\mathscr{A}(I)$ is a finite set of prime ideals. An ideal $J \subseteq I$ is said to be a *reduction of I* iff there exists a power n such that $JI^n = I^{n+1}$. The *reduction number of I* with respect to J, denoted $\mathrm{rd}_J(I)$, is the least power n making J a reduction of I. J is called a *minimal reduction* iff J contains no other reductions of I. The *reduction number* of I, denoted $\mathrm{rd}(I)$, is the least number $\mathrm{rd}_J(I)$, as J ranges over the minimal reductions of I. It is a standard fact (q.v. [NR]) that the minimal number of generators of any minimal reduction of I will always be $s(I)$ (assuming the residue field to be infinite).

We now briefly recall the concept of local cohomology. If I is an ideal in the ring R and N is a finite R-module, the i th *local cohomology module* of N with respect to I, denoted $H^i_I(N)$, is defined via the formula

$$H^i_I(N) := \varinjlim_n \mathrm{Ext}^i_R(R/I^n, N).$$

It can also be obtained as the i th cohomology of the Čech complex:

$$C^{\bullet}: \quad 0 \to N \to N_{b_1} \oplus \cdots \oplus N_{b_k} \to \cdots \to N_{b_1 \cdots b_k} \to 0,$$

where the modules are the direct sums of the various localizations, taken none at a time, one a time, two at a time, etc., and where the maps are just the sums of the canonical maps into the localizations (with appropriate minus signs), and where $(b_1, \ldots, b_k) = J$ is any ideal having the same radical as I. Note that $H^0_I(N)$ can be identified with $\bigcup_n (0 :_N I^n)$, the set of elements in N annihilated by some power of I. In the case that R and N are graded and I is homogeneous, one may take the elements $b_1, \ldots, b_k$ occurring in the complex $C^{\cdot}$ to be a set of homogeneous generators for any ideal J having the same radical as I. Consequently, the modules in the complex $C^{\cdot}$ and hence the homology modules of the complex retain a natural graded structure, and it is this which is generally referred to as *graded local cohomology*. See [HIO, Chapter VII] for more information.

We should also mention that during the preparation of this manuscript we were informed that the main result of this paper, Theorem 2.3, has been obtained independently by Aberbach-Huneke-Trung using different techniques (see [AHT]). Moreover, very closely related results have been obtained via yet another set of techniques by Simis-Ulrich-Vasconcelos (see [SUV]). Both papers should be forthcoming.

2. COMPARING CASTELNUOVO REGULARITIES

We begin by considering the vanishing in certain degrees of local cohomology modules defined with respect to different homogeneous ideals. In the proposition below we fix a d-dimensional graded Noetherian ring $\mathbf{A} = \bigoplus_{j \geq 0} A_j$. We write $B = \mathbf{A}_0$ for the degree zero component of $\mathbf{A}$ and assume that B is a local ring with maximal ideal $\mathfrak{m}$. For $p \in \mathrm{Spec}(B)$ write $\mathbf{A}_p$ for $\mathbf{A}$ localized at the multiplicatively closed set $B \backslash p$, and set $\mathfrak{P} = (p, \mathbf{A}_{p_+})\mathbf{A}_p$ and $d(p) = \dim(\mathbf{A}_p)$. For a homogeneous ideal $\mathfrak{J} \subseteq \mathbf{A}$, we will write $a_i(\mathfrak{J}, \mathbf{A})$ for the least integer k_i (including $-\infty$) such that $H^i_{\mathfrak{J}}(\mathbf{A})_j = 0$, for all $j > k_i$. When $\mathfrak{J} = \mathbf{A}_+$, $\max\{a_i(\mathbf{A}_+, \mathbf{A}) + i \mid 0 \leq i \leq d\}$ is customarily called the *Castelnuovo regularity* of $\mathbf{A}$. By abuse of terminology, we shall call $\max\{a_i(\mathfrak{J}, \mathbf{A}) + i \mid 0 \leq i \leq d\}$ the *Castelnuovo regularity of* $\mathbf{A}$ *with respect to* $\mathfrak{J}$. As mentioned above, we are interested in relating the Castelnuovo regularity of the associated graded ring of an ideal with respect to its homogeneous maximal ideal (as used in the Trung-Ikeda theorem) to the Castelnuovo regularity with respect to its positive part (as used in Trung's proposition concerning reduction number). Toward this end we offer the following proposition.

Proposition 2.1. *Let* $\mathbf{A}$ *be a graded Noetherian ring over a local ring* B. *Then*:

(i) *Let* r *be an integer such that* $H^i_{\mathfrak{P}}(\mathbf{A}_p)_j = 0$, *for all* $j \geq r$, $p \in \mathrm{Spec}(B)$, *and all* i, $0 \leq i \leq d(p)$. *Then for any homogeneous ideal* $\mathfrak{J}$ *which contains* $\mathbf{A}_+$, $H^i_{\mathfrak{J}}(\mathbf{A})_j = 0$ *for all* $j \geq r$ *and* $0 \leq i \leq d$.

(ii) *Conversely, suppose that* s *is an integer satisfying* $H^i_{\mathbf{A}_+}(\mathbf{A})_j = 0$ *for all* $j \geq s$ *and* $0 \leq i \leq d$. *Then, for all* $p \in \mathrm{Spec}(B)$ *and* $0 \leq i \leq d(p)$, $H^i_{\mathfrak{P}}(\mathbf{A}_p)_j = 0$ *for all* $j \geq s$.

(iii) *Let* t *be an integer such that* $H^{d(p)}_{\mathbf{A}_{p_+}}(\mathbf{A}_p)_j = 0$, *for all* $j \geq t$ *and* $p \in \mathrm{Spec}(B)$. *Then for any homogeneous ideal* $\mathfrak{J}$ *which contains* $\mathbf{A}_+$, $H^d_{\mathfrak{J}}(\mathbf{A})_j = 0$, *for all* $j \geq t$.

Before turning to the proof of the proposition, we make a couple of remarks concerning (graded) local cohomology.

Remarks. (a) If R is a (not necessarily graded) Noetherian ring, $x \in R$, and $I \subseteq R$ is an ideal, then according to [B, (3.9) Lemma], for any R-module M there is a long exact sequence

$$0 \to H^0_{(I,x)}(M) \to H^0_I(M) \to H^0_{I_x}(M_x) \to H^1_{(I,x)}(M) \to$$

$$\cdots \to H^i_{(I,x)}(M) \to H^i_I(M) \to H^i_{I_x}(M_x) \to H^{i+1}_{(I,x)}(M) \to \cdots .$$

If R and M are graded and I and x are homogeneous, then the maps in this sequence are *degree-preserving.*

(b) If $S \subseteq R$ is a multiplicatively-closed set then $H^i_I(M)_S \cong H^i_{IR_S}(M_S)$, i.e., local cohomology commutes with localization. If R and M are graded, I is homogeneous, and $S \subseteq R_0$, then $[H^i_I(M)_j]_S \cong [H^i_{IR_S}(M_S)]_j$ as $(R_0)_S$-modules.

(c) It follows from (b) that if R and M are graded, $x \in R_0$, and I is homogeneous, then $H^i_{I_x}(M_x)_j = 0$ if and only if for all $p \in \mathrm{Spec}(R_0)_x$, $[H^i_{IR_p}(M_p)]_j = 0$.

Proof of Proposition 2.1. To prove (i), suppose $H^i_{\mathfrak{P}}(\mathbf{A}_p)_j = 0$ for all $j \geq r$, all $p \in \mathrm{Spec}(B)$, and all i, $0 \leq i \leq d(p)$. In order to prove $H^i_{\mathfrak{J}}(\mathbf{A})_j = 0$ for all $j \geq r$ and $0 \leq i \leq d$, we proceed by induction on $\dim(\mathbf{A}/\mathfrak{J})$. If $\dim(\mathbf{A}/\mathfrak{J}) = 0$, $\mathfrak{J}$ has radical equal to the homogeneous maximal ideal of $\mathbf{A}$, and hence the same local cohomology modules, so there is nothing to prove. Now suppose $\dim(\mathbf{A}/\mathfrak{J}) > 0$. Since $\mathbf{A}_+ \subseteq \mathfrak{J}$, we may find a homogeneous element $x \in B$ of degree zero such that $\dim(\mathbf{A}/(\mathfrak{J}, x)) < \dim(\mathbf{A}/\mathfrak{J})$. Consider the "degree j level" of the Brodmann exact sequence:

$$\cdots \to H^{i-1}_{\mathfrak{J}\mathbf{A}_x}(\mathbf{A}_x)_j \to H^i_{(\mathfrak{J},x)}(\mathbf{A})_j \to H^i_{\mathfrak{J}}(\mathbf{A})_j \to H^i_{\mathfrak{J}\mathbf{A}_x}(\mathbf{A}_x)_j \to \cdots .$$

We claim that $H^i_{\mathfrak{J}\mathbf{A}_x}(\mathbf{A}_x)_j = 0$. To see this, by the remarks above it suffices to show that $[H^i_{\mathfrak{J}\mathbf{A}_p}(\mathbf{A}_p)]_j = 0$ for all $p \in \mathrm{Spec}(B_x)$. But in the ring $\mathbf{A}_p$, the dimension of $\mathbf{A}_p/\mathfrak{J}_p$ has dropped, so this module is zero by induction on the dimension of $\mathbf{A}/\mathfrak{J}$. On the other hand, we also have that $H^i_{(\mathfrak{J},x)}(\mathbf{A})_j = 0$ by our induction hypothesis. By the Brodmann exact sequence, $H^i_{\mathfrak{J}}(\mathbf{A})_j = 0$, as desired.

For the second part, we suppose that for all $i \geq 0$ and $j \geq s$, $H^i_{\mathbf{A}_+}(\mathbf{A})_j = 0$, and we seek to prove that for all $p \in \mathrm{Spec}(B)$, $i \geq 0$, and $j \geq s$, $H^i_{\mathfrak{P}}(\mathbf{A}_p)_j = 0$. This time we may proceed by induction on $\dim(B) = \dim(\mathbf{A}_0)$. If $\dim(B) = 0$, there is nothing to prove. Suppose $\dim(B) > 0$. By localizing at $p \in \mathrm{Spec}(B)$ and applying our induction hypothesis, we see it is enough to consider the case that $p = \mathfrak{m}$ is the maximal ideal of B. Now, let $\mathfrak{J}$ be maximal among the homogeneous ideals of $\mathbf{A}$ containing $\mathbf{A}_+$ such that $H^i_{\mathfrak{J}}(\mathbf{A})_j = 0$, for all i between 0 and d and all $j \geq s$. We claim $\mathfrak{J} = \mathfrak{M}$, the homogeneous maximal ideal of $\mathbf{A}$. If not, then there exist $x \in B$, $0 \leq i \leq d$, and $j \geq s$ such that $H^i_{(\mathfrak{J},x)}(\mathbf{A})_j \neq 0$. However, $H^{i-1}_{\mathfrak{J}\mathbf{A}_x}(\mathbf{A}_x)_j$ is locally zero on $\mathrm{Spec}(B_x)$ by the induction on $\dim B$, and hence zero, so we conclude from Brodmann's exact sequence that $H^i_{(\mathfrak{J},x)}(\mathbf{A})_j = 0$, a contradiction.

Finally, to prove (iii), we again induct on $\dim(B)$. If $\dim(B) = 0$, there is nothing to prove. Suppose $\dim(B) > 0$. Then $d = \dim(\mathbf{A}) \geq 1$. Let

$A_+ \subseteq \mathfrak{J}$, with $\mathfrak{J}$ homogeneous. Choose $\mathfrak{J}'$ homogeneous, $A_+ \subseteq \mathfrak{J}' \subseteq \mathfrak{J}$, and $\mathfrak{J}'$ maximal with respect to the property that $H_{\mathfrak{J}'}^d(A)_j = 0$, for all $j \geq t$. If $\mathfrak{J}' \neq \mathfrak{J}$, we may find $x \in \mathfrak{J}_0$ such that $(\mathfrak{J}', x)$ properly contains $\mathfrak{J}'$ and such that $H_{(\mathfrak{J}',x)}^d(A)_j \neq 0$, for some $j \geq t$. Consider the end of the Brodmann sequence on the jth level:

$$\cdots \to H_{\mathfrak{J}'A_x}^{d-1}(A_x)_j \to H_{(\mathfrak{J}',x)}^d(A)_j \to H_{\mathfrak{J}'}^d(A)_j \to \cdots.$$

Notice that $(H_{\mathfrak{J}'A_x}^{d-1}(A_x)_j)_p \cong [H_{\mathfrak{J}'A_p}^{d-1}(A_p)]_j$, for all $p \in \mathrm{Spec}(B_x)$, by the remarks above. For any such p, $d(p) \leq d-1$, so $[H_{\mathfrak{J}'A_p}^{d-1}(A_p)]_j = 0$ (either by induction on $\dim(B)$ or dimension-theoretic considerations). Thus $H_{\mathfrak{J}'A_x}^{d-1}(A_x)_j = 0$. From the exact sequence above, $H_{(\mathfrak{J}',x)}^d(A)_j = 0$, contradiction. $\square$

Corollary 2.2. *Let the notation be as in Proposition* 2.1. *Let* $d = \dim(A)$, *and suppose that for some integer* r, $H_{\mathfrak{P}}^i(A_p)_j = 0$ *for all* $j \geq r$, *all* $p \in \mathrm{Spec}(B)$, *and all* $0 \leq i \leq d(p)$. *Let* $\mathfrak{J}$ *be any homogeneous ideal containing* A_+. *Then the Castelnuovo regularity of* A *with respect to* $\mathfrak{J}$ *is less than* $d + r$.

Proof. Immediate from the definition of Castelnuovo regularity and Proposition 2.1. $\square$

We are now ready for our theorem.

Theorem 2.3 (Generalized Goto-Shimoda). *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring with infinite residue field and* $I \subseteq R$ *an ideal having positive height. The following statements are equivalent:*

  (i) $\mathscr{R}$ *is Cohen-Macaulay.*
  (ii) $\mathscr{G}$ *is Cohen-Macaulay and* $\mathrm{rd}_{J_P}(I_P) \leq s(I_P) - 1$, *for all prime ideals* P *containing* I *and all minimal reductions* J_P *of* I_P.
  (iii) $\mathscr{G}$ *is Cohen-Macaulay and* $\mathrm{rd}(I_P) \leq s(I_P) - 1$, *for all prime ideals* $P \in \mathscr{A}(I)$.

Proof. Suppose that $\mathscr{R}$ is Cohen-Macaulay. By Huneke's result [H, Proposition 1.1] $\mathscr{G}$ is Cohen-Macaulay. We want to show $\mathrm{rd}_{J_P}(I_P) \leq s(I_P) - 1$, for all prime ideals P containing I and all minimal reductions J_P of I_P. For this we may clearly localize at any P containing I, assume $P = \mathfrak{m}$, and show $\mathrm{rd}_J(I) \leq s(I) - 1$ for any minimal reduction J. Starting anew, we let $P \in \mathrm{Spec}(R)$ be any prime containing I and set $d(P) = \dim(R_P)$. Write $\mathfrak{P}$ for the homogeneous maximal ideal of $\mathscr{G}_P$. Since $\mathscr{R}_P$ is also Cohen-Macaulay, $a_{d(P)}(\mathfrak{P}, \mathscr{G}_P) < 0$ (by the theorem of Trung-Ikeda). Since $\mathscr{G}_P$ is Cohen-Macaulay, this is the only "a-invariant" of $\mathscr{G}_P$ to worry about, so we conclude from Proposition 2.1(i) (by taking $\mathscr{G} = A$, $\mathscr{G}_+ = \mathfrak{J}$, $p = P/I$, and $r = 0$) that $H_{\mathscr{G}_+}^i(\mathscr{G})_j = 0$ for all $j \geq 0$ and $0 \leq i \leq d$. Moreover, since $\mathscr{G}_+$ may be generated up to radical by $s(I)$ elements (namely, the images in I/I^2 of a minimal reduction for I), it follows that $H_{\mathscr{G}_+}^i(\mathscr{G}) = 0$ for $i > s(I)$. Hence the Castelnuovo regularity of $\mathscr{G}$ with respect to $\mathscr{G}_+$ is less than or equal to $s(I) - 1$, and the required bound on the reduction number $r_J(I)$ now follows immediately from Trung's proposition [T, Proposition 3.2]. Thus (i) implies (ii). Clearly (ii) implies (iii).

To see that (iii) implies (i), suppose that $\mathscr{G}$ is Cohen-Macaulay and $\mathrm{rd}(I_P) \leq s(I_P) - 1$, for all $P \in \mathscr{A}(I)$. By the Trung-Ikeda theorem, we need to show that $a_d(\mathfrak{M}, \mathscr{G}) < 0$. By taking $\mathbf{A} = \mathscr{G}$ and $\mathfrak{J} = \mathfrak{M}$ in the third part of Proposition 2.1, we see that it is enough to prove that $H_{\mathscr{G}_{P_+}}^{d(P)}(\mathscr{G}_P)_j = 0$, for all $j \geq 0$ and $P \in \mathrm{Spec}(R)$ containing I. If $P \notin \mathscr{A}(I)$, then $s(I_P) < \dim(R_P) = d(P)$, so $H_{\mathscr{G}_{P_+}}^{d(P)}(\mathscr{G}_P) = 0$. If $P \in \mathscr{A}(I)$, then $\mathrm{rd}(I_P) \leq s(I_P) - 1 = d(P) - 1$, so $H_{\mathscr{G}_{P_+}}^{d(P)}(\mathscr{G}_P)_j = 0$, for all $j \geq 0$ (by Trung's proposition), and the proof is complete. $\square$

Corollary 2.4 [GHO], [GH]. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field and let $I \subseteq R$ be an ideal having height at least one. Assume either that I is equimultiple or that I is generically a complete intersection with analytic deviation one. Let $s = s(I)$. Then $\mathscr{R}$ is Cohen-Macaulay if and only if $\mathscr{G}$ is Cohen-Macaulay and $\mathrm{rd}_J(I) \leq s - 1$ for every minimal reduction J of I.*

Proof. Immediate from Theorem 2.3 and the definitions. $\square$

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