Journal of Pure and Applied Algebra 59 (1989) 179–186
North-Holland

PRIME DIVISORS AND DIVISORIAL IDEALS

Daniel KATZ*
Department of Mathematics, University of Kansas, Lawrence, KS 66045, U.S.A.

Stephen McADAM
Department of Mathematics, University of Texas, Austin, TX 78712, U.S.A.

L.J. RATLIFF, Jr.**
Department of Mathematics, University of California, Riverside, CA 92521, U.S.A.

Communicated by C.A. Weibel
Received 21 September 1987
Revised 27 February 1988

Let $I_1, \dots, I_g$ be regular ideals in a Noetherian ring R. Then it is shown that there exist positive integers $k_1, \dots, k_g$ such that $(I_1^{n_1+m_1} \dots I_g^{n_g+m_g}) : (I_1^{m_1} \dots I_g^{m_g}) = I_1^{n_1} \dots I_g^{n_g}$ for all $n_i \geq k_i$ $(i = 1, \dots, g)$ and for all nonnegative integers $m_1, \dots, m_g$. Using this, it is shown that if Δ is a multiplicatively closed set of nonzero ideals of R that satisfies certain hypotheses, then the sets $\mathrm{Ass}(R/(I_1^{n_1} \dots I_g^{n_g}))$ are equal for all large positive integers $n_1, \dots, n_g$. Also, if R is locally analytically unramified, then some related results for general sets Δ are proved.

Introduction

Let R be a Noetherian ring. It is known that if J is an ideal of R, then the two sequences of sets $\mathrm{Ass}\,R/J$, $\mathrm{Ass}\,R/J^2, \dots$ and $\mathrm{Ass}\,R/J_a$, $\mathrm{Ass}\,R/(J^2)_a, \dots$ eventually stabilize to sets denoted $A^*(I)$ and $\bar{A}^*(I)$ respectively (see [2, Corollary 1.5 and Proposition 3.4]). Here J_a denotes the integral closure of J. In Section 1 these results are extended in two directions. It is shown that if $I_1, \dots, I_g$ are (regular) ideals of R and Δ is a multiplicatively closed set of ideals satisfying certain hypotheses, then asymptotic stability holds for the sets $\mathrm{Ass}\,R/(I_1^{n_1} \dots I_g^{n_g})_\Delta$, where $n_1, \dots, n_g \in \mathbb{N}$ and J_Δ is the Δ-closure of an ideal J (see below). For appropriate choices of Δ one concludes that the sets $\mathrm{Ass}\,R/I_1^{n_1} \dots I_g^{n_g}$ and $\mathrm{Ass}\,R/(I_1^{n_1} \dots I_g^{n_g})_a$ enjoy asymptotic stability. In Section 2 we consider the situation for general Δ-closures under the hypothesis that R is locally analytically unramified.

* The first author was partially supported by the General Research Fund at the University of Kansas.
** The third author was supported in part by the National Science Foundation, grant DMS-8521058.

1. Asymptotic stability of Ass $R/I_1^{n_1}\dots I_g^{n_g}$

We begin by fixing some notation.

Notation. Throughout R will be a Noetherian ring, g a fixed positive integer and $I_1,\dots,I_g$ ideals of R. $\mathbb{N}_g$ will be the set of all g-tuples of non-negative integers. If $\boldsymbol{n}=(n_1,\dots,n_g)\in\mathbb{N}_g$, then by $I^{\boldsymbol{n}}$ we mean $I_1^{n_1}\dots I_g^{n_g}$. For $1\le i\le g$, $\boldsymbol{n}(i)$ will refer to n_i, the ith component of $\boldsymbol{n}$. Also, we will write $\boldsymbol{n}\ge\boldsymbol{m}$ (respectively $\boldsymbol{n}>\boldsymbol{m}$) if $\boldsymbol{n}(i)\ge\boldsymbol{m}(i)$ (respectively, $\boldsymbol{n}(i)>\boldsymbol{m}(i)$) for all $1\le i\le g$. If $\boldsymbol{n}$ and $\boldsymbol{m}$ are in $\mathbb{N}_g$ and $h\ge 0$ is an integer, then $h\boldsymbol{n}$ and $\boldsymbol{n}\pm\boldsymbol{m}$ will be defined in the usual component-wise manner ($\boldsymbol{n}-\boldsymbol{m}$ only being defined when $\boldsymbol{n}\ge\boldsymbol{m}$). We shall denote by J_{a} the integral closure of an ideal J and by J^* the eventual stable value of $(J^2:J)\subseteq(J^3:J^2)\subseteq\cdots$. J^* was introduced in [5], and in [2, Lemma 8.2] it is shown that if J is a regular ideal, then $(J^n)^*=J^n$ for n large. Both of these operations are special cases of a more general operation, the so-called Δ-closure operation, introduced by the third author in [4].

Definition. Let J be an ideal in R and Δ a multiplicatively closed set of non-zero ideals of R. The ascending chain condition guarantees that the set $\{(JK:K)\mid K\in\Delta\}$ has maximal elements, and since for K and L in Δ, $(JKL:KL)$ contains both $(JK:K)$ and $(JL:L)$, we see that the set under consideration in fact contains a unique maximal element. Let J_Δ denote that unique maximal element. The following lemma shows that the notion of Δ-closure allows one to discuss simultaneously the asymptotic behavior of Ass R/J^n and Ass $R/(J^n)_{\mathrm{a}}$:

1.1. Lemma. *Let Δ be a multiplicatively closed set of non-zero ideals.*

(a) *If every ideal in Δ is regular, then for any ideal J, $J_\Delta\subseteq J_{\mathrm{a}}$.*

(b) *If Δ equals the set of all regular ideals and J is regular, $J_\Delta=J_{\mathrm{a}}$.*

(c) *If J is a regular ideal and $\Delta=\{J^n\mid n\in\mathbb{N}\}$, then $(J^n)_\Delta=(J^n)^*$ for all n and $(J^n)_\Delta=(J^n)^*=J^n$ for all large n.*

Proof. The proofs are easy, but we include them for the convenience of the reader. For (a), $J_\Delta=(JK:K)$ for some $K\in\Delta$. Suppose K is generated by $k_1,\dots,k_n$. Then for $x\in J_\Delta$ and $1\le i\le n$ we have $x\cdot k_i=\sum_{j=1}^n a_{ij}k_j$ for $a_{ij}\in J$. Now a standard determinant argument shows $x\in J_{\mathrm{a}}$. For (b), suppose Δ is the set of all regular ideals and $J_\Delta=(JK:K)$ for some $K\in\Delta$. Let $x\in J_{\mathrm{a}}$. Then $J(J,x)^n=(J,x)^{n+1}$ for some n. Thus $x(J,x)^n\subseteq J(J,x)^n$, so $xK(J,x)^n\subseteq JK(J,x)^n$. Since $(J,x)\in\Delta$, it follows that $J_\Delta=(JK(J,x)^n:K(J,x)^n)$, so $x\in J_\Delta$. Thus $J_{\mathrm{a}}\subseteq J_\Delta$ and equality holds by part (a). For (c), let J be a regular ideal and $\Delta=\{J^n\mid n\in\mathbb{N}\}$. Then $(J^n)^*=((J^n)^{h+1}:(J^n)^h)$ for large h. Thus $(J^n)^*=(J^n(J^{nh}):J^{nh})\subseteq(J^n)_\Delta$. On the other hand, $(J^n)_\Delta=(J^nJ^k:J^k)$ for some k, so $(J^n)_\Delta=(J^{n+k}:J^k)\subseteq((J^n)^{k+1}:(J^n)^k)\subseteq(J^n)^*$. Thus $(J^n)_\Delta=(J^n)^*$ and the second part of (c) follows from [2, Lemma 8.2]. $\square$

Ideals of the form $(J^{n+1}:J)$ play a vital role in discussing the behavior of various prime divisors associated to large powers of J. The following lemma and proposition will play analogous roles in determining the corresponding behavior of the prime divisors associated to the product of large powers of $I_1,\dots,I_g$. In fact, we consider part (c) of Proposition 1.4 to be one of the main results of this paper.

1.2. Lemma. *Let $I_1,\dots,I_g$ be regular ideals.*

(a) *Suppose $\boldsymbol{n}$ and $\boldsymbol{m}$ are in $\mathbb{N}_g$ with $\boldsymbol{n}\geq(1,\dots,1)$. Let k be an integer with $k\boldsymbol{n}\geq\boldsymbol{m}$. Then $(I^{\boldsymbol{n}+\boldsymbol{m}}:I^{\boldsymbol{m}})\subseteq((I^{\boldsymbol{n}})^{k+1}:(I^{\boldsymbol{n}})^k)\subseteq(I^{\boldsymbol{n}})^*$.*

(b) *If we set $\Delta=\{I^{\boldsymbol{m}}\mid \boldsymbol{m}\in\mathbb{N}_g\}$, then for $\boldsymbol{n}\geq(1,\dots,1)$, $(I^{\boldsymbol{n}})^*=(I^{\boldsymbol{n}})_\Delta$.*

Proof. For (a), suppose $x\in(I^{\boldsymbol{n}+\boldsymbol{m}}:I^{\boldsymbol{m}})$. Since $k\boldsymbol{n}-\boldsymbol{m}\in\mathbb{N}_g$, we may write $(I^{\boldsymbol{n}})^k=I^{\boldsymbol{m}}I^{k\boldsymbol{n}-\boldsymbol{m}}$. Thus $x(I^{\boldsymbol{n}})^k=xI^{\boldsymbol{m}}I^{k\boldsymbol{n}-\boldsymbol{m}}\subseteq I^{\boldsymbol{n}+\boldsymbol{m}}I^{k\boldsymbol{n}-\boldsymbol{m}}=(I^{\boldsymbol{n}})^{k+1}$. This gives the first containment of the conclusion. The second containment is by the definition of $(I^{\boldsymbol{n}})^*$. For (b), suppose $\Delta=\{I^{\boldsymbol{m}}\mid \boldsymbol{m}\in\mathbb{N}_g\}$ and $\boldsymbol{n}\geq(1,\dots,1)$. Then for large integers h, $(I^{\boldsymbol{n}})^*=((I^{\boldsymbol{n}})^{h+1}:(I^{\boldsymbol{n}})^h)=(I^{\boldsymbol{n}}I^{h\boldsymbol{n}}:I^{h\boldsymbol{n}})\subseteq(I^{\boldsymbol{n}})_\Delta$, by the definition of $(I^{\boldsymbol{n}})_\Delta$. For the reverse inclusion, there is an $\boldsymbol{m}\in\mathbb{N}_g$ with $(I^{\boldsymbol{n}})_\Delta=(I^{\boldsymbol{n}+\boldsymbol{m}}:I^{\boldsymbol{m}})$. By the first part of the lemma, this last ideal is contained in $(I^{\boldsymbol{n}})^*$. □

1.3. Remark. (a) Note that $k=\max\{\boldsymbol{m}(i)\mid 1\leq i\leq g\}$ satisfies the hypothesis of Lemma 1.2(a).

(b) In Lemma 1.2, if we do not have $\boldsymbol{n}\geq(1,\dots,1)$, we cannot be assured that $(I^{\boldsymbol{n}+\boldsymbol{m}}:I^{\boldsymbol{m}})\subseteq(I^{\boldsymbol{n}})^*$. By [5, (3.4) and (4.2)], there exist regular ideals I_1 and I_2 with I_1^* properly contained in $(I_1I_2:I_2)$. Let $\boldsymbol{n}=(1,0)$ and $\boldsymbol{m}=(0,1)$. Then $(I^{\boldsymbol{n}+\boldsymbol{m}}:I^{\boldsymbol{m}})=(I_1I_2:I_2)\not\subseteq I_1^*=(I^{\boldsymbol{n}})^*$.

1.4. Proposition. *Let $I_1,\dots,I_g$ be ideals of R. Fix $1\leq i\leq g$. For each $\boldsymbol{s}\in\mathbb{N}_{g-1}$ write $J^{\boldsymbol{s}}$ for $I_1^{s_1}\dots I_{i-1}^{s_{i-1}}I_{i+1}^{s_i}\dots I_g^{s_{g-1}}$.*

(a) *For a finitely generated R module M and submodule $N\subseteq M$, there exists $k_i\in\mathbb{N}$ such that for all $n_i\geq k_i$, $I_i^{n_i}J^{\boldsymbol{s}}M\cap N=I_i^{n_i-k_i}(I_i^{k_i}J^{\boldsymbol{s}}M\cap N)$ for all $\boldsymbol{s}\in\mathbb{N}_{g-1}$.*

(b) *There exists $l_i\in\mathbb{N}$ such that $(I_i^{h+n_i}J^{\boldsymbol{s}}:I_i^h)\cap I_i^{l_i}J^{\boldsymbol{s}}=I_i^{n_i}J^{\boldsymbol{s}}$ for all $n_i>l_i$, $\boldsymbol{s}\in\mathbb{N}_{g-1}$ and $h\in\mathbb{N}$.*

(c) *If I_i is a regular ideal, there exists $d_i\in\mathbb{N}$ such that $(I_i^{h+n_i}J^{\boldsymbol{s}}:I_i^h)=I_i^{n_i}J^{\boldsymbol{s}}$ for all $n_i>d_i$, $h\in\mathbb{N}$ and $\boldsymbol{s}\in\mathbb{N}_g$. Consequently, there exists $\boldsymbol{k}\in\mathbb{N}_g$ such that $(I^{\boldsymbol{n}+\boldsymbol{m}}:I^{\boldsymbol{m}})=I^{\boldsymbol{n}}$ for all $\boldsymbol{n}>\boldsymbol{k}$ and $\boldsymbol{m}\in\mathbb{N}_g$ (if each I_i is regular).*

Proof. Let $t_1,\dots,t_g$ be indeterminates and set $\mathcal{R}=R[I_1t_1,\dots,I_gt_g]$, the Rees ring of R with respect to $I_1,\dots,I_g$. Let $\mathcal{M}=\mathcal{R}\otimes_R M$ and $\mathcal{N}$ be the submodule consisting of all finite sums of the form $\sum a_{\boldsymbol{r}}t^{\boldsymbol{r}}$ where $a_{\boldsymbol{r}}\in I^{\boldsymbol{r}}M\cap N$ (here we are writing $t^{\boldsymbol{r}}$ for $t_1^{r_1}\dots t_g^{r_g}$ if $\boldsymbol{r}\in\mathbb{N}_g$). Then $\mathcal{M}$ is an $\mathbb{N}_g$-graded finitely generated $\mathcal{R}$-module and $\mathcal{N}$ has a system of homogeneous generators. As in the proof of the usual Artin–Rees Lemma, let k_i be the maximum value achieved by any exponent of t_i in any one of the generators. Then it is readily seen that the conclusion of (a) holds for this k_i.

For (b) let $\mathscr{B}=(I_i\mathscr{R}:I_it_i)$ in $\mathscr{R}$. A brief computation shows that $\mathscr{B}$ is an $\mathbb{N}_g$-homogeneous $\mathscr{R}$-ideal, so it has a generating set of the form $a_1t^{r_1},\dots,a_st^{r_s}$, where $r_j\in\mathbb{N}_g$ and $a_j\in I^{r_j}$. Let $l_i=\{\max r_j(i)\mid 1\le j\le s\}+1$ and suppose $ct^r\in\mathscr{B}$ satisfies $r(i)>l_i$.

We may write $ct^r=\sum_j (b_jt^{r-r_j})(a_jt^{r_j})$ for elements $b_jt^{r-r_j}\in\mathscr{R}$. The choice of r forces each $b_jt^{r-r_j}\in(I_it_i)\mathscr{R}$ so $ct^r\in I_i\mathscr{R}$.

Now suppose $n_i\in\mathbb{N}$ satisfies $n_i>l_i$. Let $s\in\mathbb{N}_{g-1}$ and suppose $cI_i\subseteq I^{n_i+1}J^s$, for $c\in I_i^{l_i}J^s$. Then, writing t^s for $t_1^{s_1}\dots t_{i-1}^{s_{i-1}}t_{i+1}^{s_i}\dots t_g^{s_{g-1}}$ we have $(ct_i^{l_i}t^s)(I_it_i)\subseteq I^{n_1+1}J^st_i^{l_i+1}t^s\subseteq I_i\mathscr{R}$ (since $n_i>l_i$). By the preceding paragraph, $ct_i^{l_i}t^s\in I_i\mathscr{R}$ so $c\in I_i^{l_i+1}J^s$. We may now repeat the argument until $c\in I_i^{n_i}J^s$ as desired. This shows $(I_i^{1+n_i}J^s:I_i)\cap I_i^{l_i}J^s=I_i^{n_i}J^s$, and the rest of (b) follows from this. To finish, let $a_1,\dots,a_s$ be a set of regular elements generating I_i. As in the proof of [3, Proposition 11(e)] set $M=R\cdot(1/a_1)\oplus\cdots\oplus R\cdot(1/a_s)$ (considered as a submodule of $K\oplus\cdots\oplus K$, for K the total quotient ring of R) and $N=\{(r/1,\dots,r/1)\mid r\in R\}$. From part (a) there is $k_i\in\mathbb{N}$ such that $I_i^{n_i}J^sM\cap N=I_i^{n_i-k_i}(I_i^{k_i}J^sM\cap N)$ for all $n_i\ge k_i$, and $s\in\mathbb{N}_{g-1}$. It follows readily that $(I_i^{n_i}J^s:I_i)=I_i^{n_i-k_i}(I_i^{k_i}J^s:I_i)\subseteq I_i^{k_i}J^s$, for $n_i>k_i$. Since we may increase k_i so that it is larger than l_i, for l_i as in part (b), it follows that $(I_i^{n_i+h}J^s:I_i^h)=I_i^{n_i}J^s$ for all large n_i, $h\in\mathbb{N}$ and $s\in\mathbb{N}_{g-1}$. The second statement follows from this. □

1.5. Corollary. *Let $I_1,\dots,I_g$ be regular ideals. There is a $\boldsymbol{d}\in\mathbb{N}_g$ such that for all $\boldsymbol{n}\in\mathbb{N}_g$ with $\boldsymbol{n}\ge\boldsymbol{d}$, $(I^{\boldsymbol{n}})^*=I^{\boldsymbol{n}}$.*

Proof. Let $\boldsymbol{k}$ be as in Proposition 1.4(c) so that $(I^{\boldsymbol{n}+\boldsymbol{m}}:I^{\boldsymbol{m}})=I^{\boldsymbol{m}}$ for all $\boldsymbol{n}\ge\boldsymbol{k}$, $\boldsymbol{m}\in\mathbb{N}_g$ and let $\boldsymbol{d}$ be such that $\boldsymbol{d}(i)=\max\{1,\boldsymbol{k}(i)\}$ for $1\le i\le g$. The corollary now follows from Proposition 1.4(c) and Lemma 1.2(b). □

1.6. Proposition. (a) *The set* $\bigcup\{\operatorname{Ass}R/I^{\boldsymbol{n}}\mid\boldsymbol{n}\in\mathbb{N}_g\}$ *is finite.*

(b) $\bigcup\{\operatorname{Ass}R/(I^{\boldsymbol{m}})_{\mathrm{a}}\mid\boldsymbol{m}\in\mathbb{N}_g\}\subseteq\bigcup\{\operatorname{Ass}R/(I^{\boldsymbol{n}})\mid\boldsymbol{n}\in\mathbb{N}_g\}$.

(c) *If* $\Delta\subseteq\{I^{\boldsymbol{m}}\mid\boldsymbol{m}\in\mathbb{N}_g\}$, *then* $\bigcup\{\operatorname{Ass}R/(I^{\boldsymbol{m}})_\Delta\mid\boldsymbol{m}\in\mathbb{N}_g\}\subseteq\bigcup\{\operatorname{Ass}R/(I^{\boldsymbol{n}})\mid\boldsymbol{n}\in\mathbb{N}_g\}$.

(d) *If $I_1,\dots,I_g$ are regular ideals, then* $\bigcup\{\operatorname{Ass}R/(I^{\boldsymbol{m}})^*\mid\boldsymbol{m}\in\mathbb{N}_g\}\subseteq\bigcup\{\operatorname{Ass}R/(I^{\boldsymbol{n}})\mid\boldsymbol{n}\in\mathbb{N}_g\}$.

Proof. Let $\mathscr{S}=R[I_1t_1,\dots,I_gt_g,t_1^{-1},\dots,t_g^{-1}]$ be the extended Rees ring of R with respect to $I_1,\dots,I_g$ and set $u_i=t_i^{-1}$. For $\boldsymbol{n}\in\mathbb{N}_g$, $u^{\boldsymbol{n}}\mathscr{S}\cap R=I^{\boldsymbol{n}}$. Thus any $P\in\operatorname{Ass}R/I^{\boldsymbol{n}}$ lifts to a prime divisor $\mathscr{P}$ of $\mathscr{S}/u^{\boldsymbol{n}}\mathscr{S}$. For some $1\le i\le g$, $u_i\in\mathscr{P}$ and because each u_i is regular, $\mathscr{P}$ must be a prime divisor of $u_i\mathscr{S}$. Now $\bigcup\{\operatorname{Ass}\mathscr{S}/u^{\boldsymbol{n}}\mathscr{S}\mid\boldsymbol{n}\in\mathbb{N}_g\}$ is finite, so $\bigcup\{\operatorname{Ass}R/I^{\boldsymbol{n}}\mid\boldsymbol{n}\in\mathbb{N}_g\}$ is finite as well. Thus we have (a).

For (b), by [2, Propositions 3.9 and 3.17], $\operatorname{Ass}R/(I^{\boldsymbol{n}})_{\mathrm{a}}\subseteq A^*(I^{\boldsymbol{m}})=\operatorname{Ass}R/I^{h\boldsymbol{m}}$ for all large integers h. Hence part (b).

For (c), let $\mathbf{m} \in \mathbb{N}_g$ and $P \in \operatorname{Ass} R/(I^{\mathbf{m}})_\Delta$. We may write $P = ((I^{\mathbf{m}})_\Delta : x) = ((I^{\mathbf{m}} I^{\mathbf{k}} : I^{\mathbf{k}}) : x) = (I^{\mathbf{m}+\mathbf{k}} : x I^{\mathbf{k}})$, by the definition of Δ. Hence $P \in \operatorname{Ass} R/I^{\mathbf{m}+\mathbf{k}}$ and (c) follows.

For (d), we must show $\operatorname{Ass} R/(I^{\mathbf{m}})^* \subseteq \bigcup\{\operatorname{Ass} R/I^{\mathbf{n}} \mid \mathbf{n} \in \mathbb{N}_g\}$. Clearly it does no harm to assume $\mathbf{m} \geq (1, \ldots, 1)$ (since zero components can simply be ignored). Let $\Delta = \{I^{\mathbf{n}} \mid \mathbf{n} \in \mathbb{N}_g\}$. By Lemma 1.2(b), $\operatorname{Ass} R/(I^{\mathbf{m}})^* = \operatorname{Ass} R/(I^{\mathbf{m}})_\Delta \subseteq \bigcup\{\operatorname{Ass} R/I^{\mathbf{n}} \mid \mathbf{n} \in \mathbb{N}_g\}$ by (c). □

1.7. Theorem. *Let Δ be a multiplicatively closed set of non-zero ideals with $\{I^{\mathbf{n}} \mid \mathbf{n} \in \mathbb{N}_g\} \subseteq \Delta$. Then*

(a) *For any $\mathbf{n}, \mathbf{k} \in \mathbb{N}_g$ satisfying $\mathbf{n} \geq \mathbf{k}$, $\operatorname{Ass} R/(I^{\mathbf{k}})_\Delta \subseteq \operatorname{Ass} R/(I^{\mathbf{n}})_\Delta$.*

(b) *If $\bigcup\{\operatorname{Ass} R/(I^{\mathbf{m}})_\Delta \mid \mathbf{m} \in \mathbb{N}_g\} \subseteq \bigcup\{\operatorname{Ass} R/(I^{\mathbf{n}}) \mid \mathbf{n} \in \mathbb{N}_g\}$, then for any sequence $\mathbf{n}_1 \leq \mathbf{n}_2 \leq \cdots$ of elements from $\mathbb{N}_g$, the sequence $\operatorname{Ass} R/(I^{\mathbf{n}_1})_\Delta \subseteq \operatorname{Ass} R/(I^{\mathbf{n}_2})_\Delta \subseteq \cdots$ eventually stabilizes. In particular, there exists $\mathbf{k} \in \mathbb{N}_g$ such that $\operatorname{Ass} R(I^{\mathbf{n}})_\Delta$ is independent of $\mathbf{n}$, for all $\mathbf{n} \geq \mathbf{k}$.*

Proof. For (a), let $P = ((I^{\mathbf{k}})_\Delta : x)$ belong to $\operatorname{Ass} R/(I^{\mathbf{k}})_\Delta$, with $x \in R$. Writing $(I^{\mathbf{k}})_\Delta = (I^{\mathbf{k}} K : K)$ for some $K \in \Delta$, we see that $I^{\mathbf{n}-\mathbf{k}}(I^{\mathbf{k}})_\Delta = I^{\mathbf{n}-\mathbf{k}}(I^{\mathbf{k}} K : K) \subseteq (I^{\mathbf{n}} K : K) \subseteq (I^{\mathbf{n}})_\Delta$. Thus $P = ((I^{\mathbf{k}})_\Delta : x) \subseteq (I^{\mathbf{n}-\mathbf{k}}(I^{\mathbf{k}})_\Delta : x I^{\mathbf{n}-\mathbf{k}}) \subseteq ((I^{\mathbf{n}})_\Delta : x I^{\mathbf{n}-\mathbf{k}})$. However, we also claim that this last ideal is contained in P. Let y belong to this ideal. Then $yx \in ((I^{\mathbf{n}})_\Delta : I^{\mathbf{n}-\mathbf{k}})$. For some $L \in \Delta$, $(I^{\mathbf{n}})_\Delta = (I^{\mathbf{n}} L : L)$, so $yx \in ((I^{\mathbf{n}} L : L) : I^{\mathbf{n}-\mathbf{k}}) = (I^{\mathbf{k}} I^{\mathbf{n}-\mathbf{k}} L : I^{\mathbf{n}-\mathbf{k}} L) \subseteq (I^{\mathbf{k}})_\Delta$ since $I^{\mathbf{n}-\mathbf{k}} L \in \Delta$. Therefore $y \in ((I^{\mathbf{k}})_\Delta : x) = P$ as desired. Thus $P = ((I^{\mathbf{n}})_\Delta : x I^{\mathbf{n}-\mathbf{k}})$, so $P \in \operatorname{Ass} R/(I^{\mathbf{n}})_\Delta$.

For (b), by Proposition 1.6 we have that $\bigcup\{\operatorname{Ass} R/(I^{\mathbf{n}})_\Delta \mid \mathbf{n} \in \mathbb{N}_g\}$ is finite, so if $\mathbf{n}_1 \leq \mathbf{n}_2 \leq \cdots$, then by (a), $\operatorname{Ass} R/(I^{\mathbf{n}_1})_\Delta \subseteq \operatorname{Ass} R, (I^{\mathbf{n}_2})_\Delta \subseteq \cdots$ and this sequence must eventually stabilize. Now suppose that $\mathbf{k} = (k, \ldots, k) \in \mathbb{N}_g$ is such that $\operatorname{Ass} R/(I^{\mathbf{k}})_\Delta = \operatorname{Ass} R/(I^{h\mathbf{k}})_\Delta$ for all $h \in \mathbb{N}$. (This follows from the $g = 1$ case of what was just shown.) For $\mathbf{n} \geq \mathbf{k}$, select $h \in \mathbb{N}$ such that $h\mathbf{k} \geq \mathbf{n} \geq \mathbf{k}$. Then by part (a), $\operatorname{Ass} R/(I^{\mathbf{k}})_\Delta \subseteq \operatorname{Ass} R/(I^{\mathbf{n}})_\Delta \subseteq \operatorname{Ass} R/(I^{h\mathbf{k}})_\Delta = \operatorname{Ass} R/(I^{\mathbf{k}})_\Delta$. □

1.8. Corollary. Let $I_1, \ldots, I_g$ be regular ideals.

(a) *If $\mathbf{n}_1 \leq \mathbf{n}_2 \leq \cdots$ is an increasing sequence from $\mathbb{N}_g$, then the sequence $\operatorname{Ass} R/(I^{\mathbf{n}_1})_{\mathrm{a}} \subseteq \operatorname{Ass} R/(I^{\mathbf{n}_2})_{\mathrm{a}} \subseteq \cdots$ eventually stabilizes. In particular, there exists $\mathbf{k} \in \mathbb{N}_g$ such that $\operatorname{Ass} R/(I^{\mathbf{n}})_{\mathrm{a}}$ is independent of $\mathbf{n}$ for all $\mathbf{n} \geq \mathbf{k}$.*

(b) *A similar statement holds for $\operatorname{Ass} R/(I^{\mathbf{n}})^*$, provided $\mathbf{n}_1 \geq (1, \ldots, 1)$.*

(c) *Let $\mathbf{n}_1 \leq \mathbf{n}_2 \leq \cdots$ be an increasing sequence from $\mathbb{N}_g$. Then the sequence $\operatorname{Ass} R/I^{\mathbf{n}_1}$, $\operatorname{Ass} R/I^{\mathbf{n}_2}$, $\cdots$ eventually stabilizes. In particular, there exists $\mathbf{k} \in \mathbb{N}_g$ such that $\operatorname{Ass} R/I^{\mathbf{n}}$ is independent of $\mathbf{n}$ for $\mathbf{n} \geq \mathbf{k}$.*

Proof. (a) follows from 1.1(b), 1.6 and 1.7 while (b) follows from 1.2(b), 1.6 and 1.7. For (c), we may suppose that for $1 \leq i \leq k$, $\{\mathbf{n}_j(i) \mid j \geq 1\}$ is infinite and for

$k+1 \le i \le g$, $\{\boldsymbol{n}_j(i) \mid j \ge 1\}$ is finite. By ignoring small values of j we may assume that $(\boldsymbol{n}_j(k+1), \dots, \boldsymbol{n}_j(g)) = (s_1, \dots, s_{g-k}) = \boldsymbol{s} \in \mathbb{N}_{g-k}$. Let $\boldsymbol{t}_j \in \mathbb{N}_k$ be such that $\boldsymbol{n}_j = (t_j(1), \dots, t_j(k), s_1, \dots, s_{g-k})$ and write $I^{\boldsymbol{n}_j} = A^{t_j} B^{s}$, where $A^{t_j} = I_1^{t_j(1)} \dots I_k^{t_j(k)}$ and $B^s = I_{k+1}^{s_1} \dots I_g^{s_{g-k}}$. Let $\Delta = \{A^t \mid t \in \mathbb{N}_k\}$. Arguing as in the proof of 1.7(a), it is readily seen that $\operatorname{Ass} R/(I^{\boldsymbol{n}_1})_\Delta \subseteq \operatorname{Ass} R/(I^{\boldsymbol{n}_2})_\Delta \subseteq \cdots$. On the other hand, $(I^{\boldsymbol{n}_j})_\Delta = (A^{t_j} B^s)_\Delta$ has the form $(A^{t_j} B^s A^r : A^r) = A^{t_j} B^s$ for j large (by Proposition 1.4). Thus $(I^{\boldsymbol{n}_j})_\Delta = (I^{\boldsymbol{n}_j})$ for j large and part (c) now follows from 1.6 and 1.7. □

2. The locally analytically unramified case

In this section we show that if R is locally analytically unramified with finite integral closure, then $\operatorname{Ass} R/(I^{\boldsymbol{n}})_\Delta$ enjoys asymptotic stability for very general Δ-closures. We also show that there exists a single $K \in \Delta$ satisfying $(I^{\boldsymbol{n}})_\Delta = (I^{\boldsymbol{n}} K : K)$ for all $\boldsymbol{n} \in \mathbb{N}_g$. This is accomplished by proving the following variation of the Artin–Rees lemma:

2.1. Lemma. *Let $I_1, \dots, I_g$ be ideals of R. For indeterminates $t_1, \dots, t_g$ set $\mathcal{R} = R[It_1, \dots, I_g t_g]$ and $\mathcal{R}_\Delta = R[\{(I^{\boldsymbol{n}})_\Delta t^{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{N}_g\}]$. (Note that $(I^{\boldsymbol{n}})_\Delta \cdot (I^{\boldsymbol{m}})_\Delta \subseteq (I^{\boldsymbol{n}+\boldsymbol{m}})_\Delta$, so $\mathcal{R}_\Delta$ is a ring and also an $\mathcal{R}$-module.) Then*

(a) *If $\mathcal{R}_\Delta$ is a finite $\mathcal{R}$-module, there exists $K \in \Delta$ such that for all $\boldsymbol{n} \in \mathbb{N}_g$, $(I^{\boldsymbol{n}})_\Delta = (I^{\boldsymbol{n}} K : K)$. Also, there is an integer b such that if $\boldsymbol{n}$ and $\boldsymbol{m}$ are such that for all $1 \le i \le g$ either $\boldsymbol{n}(i) = \boldsymbol{m}(i)$ or $\boldsymbol{n}(i) \ge \boldsymbol{m}(i) \ge b$, then $(I^{\boldsymbol{n}})_\Delta = I^{\boldsymbol{n}-\boldsymbol{m}} (I^{\boldsymbol{m}})_\Delta$. In particular, if $\boldsymbol{n} \ge \boldsymbol{m} \ge (b, \dots, b)$, then $(I^{\boldsymbol{n}})_\Delta = I^{\boldsymbol{n}-\boldsymbol{m}} (I^{\boldsymbol{m}})_\Delta$.*

(b) *If there is a regular ideal $K \in \Delta$ such that $(I^{\boldsymbol{n}})_\Delta = (I^{\boldsymbol{n}} K : K)$ for all $\boldsymbol{n} \in \mathbb{N}_g$, then $\mathcal{R}_\Delta$ is a finite $\mathcal{R}$-module.*

Proof. For (a), the hypothesis implies that there exist finitely many $\boldsymbol{m}_j \in \mathbb{N}_g$ such that $\mathcal{R}_\Delta = \sum \mathcal{R}((I^{\boldsymbol{m}_j})_\Delta t^{\boldsymbol{m}_j})$ over $1 \le j \le r$.

For each j, there is a $K_j \in \Delta$ such that $(I^{\boldsymbol{m}_j})_\Delta = (I^{\boldsymbol{m}_j} K_j : K_j)$. Let K be the product of the K_j over all $1 \le j \le r$. Then $(I^{\boldsymbol{m}_j})_\Delta = (I^{\boldsymbol{m}_j} K : K)$ for all j.

Now, consider the submodule $\mathcal{T}$ of $\mathcal{R}_\Delta$ having the form $\sum (I^{\boldsymbol{n}} K : K) t^{\boldsymbol{n}}$ over all $\boldsymbol{n} \in \mathbb{N}_g$. (Since for $\boldsymbol{m} \in \mathbb{N}_g$, $I^{\boldsymbol{m}}(I^{\boldsymbol{n}} K : K) \subseteq (I^{\boldsymbol{n}+\boldsymbol{m}} K : K)$, this is a submodule.) Since for $1 \le j \le r$, $\mathcal{T}$ contains $(I^{\boldsymbol{m}_j} K : K) t^{\boldsymbol{m}} = (I^{\boldsymbol{m}_j})_\Delta t^{\boldsymbol{m}}$, and these last sets generate $\mathcal{R}_\Delta$ over $\mathcal{R}$ as an $\mathcal{R}$-module, we see that $\mathcal{T} = \mathcal{R}_\Delta$. It follows that $(I^{\boldsymbol{n}} K : K) = (I^{\boldsymbol{n}})_\Delta$ for all $\boldsymbol{n} \in \mathbb{N}_g$. This proves the first part of (a).

Now let $b = \max\{\boldsymbol{m}_j(i) \mid 1 \le j \le r,\ 1 \le i \le g\}$. Suppose that $\boldsymbol{n}$ and $\boldsymbol{m}$ are such that for each $1 \le i \le g$, either $\boldsymbol{n}(i) = \boldsymbol{m}(i)$ or $\boldsymbol{n}(i) \ge \boldsymbol{m}(i) \ge b$. Since $\mathcal{R}_\Delta = \sum \mathcal{R}((I^{\boldsymbol{m}_j})_\Delta t^{\boldsymbol{m}_j})$ over $1 \le j \le r$, looking at the $t^{\boldsymbol{n}}$th term in $\mathcal{R}_\Delta$, we see that $(I^{\boldsymbol{n}})_\Delta = \sum (I^{\boldsymbol{n}-\boldsymbol{m}_j})(I^{\boldsymbol{m}_j})_\Delta$ over those $1 \le j \le r$ with $\boldsymbol{m}_j \le \boldsymbol{n}$. A similar statement can be made about $(I^{\boldsymbol{m}})_\Delta$. However, we claim that $\boldsymbol{m}_j \le \boldsymbol{n}$ if and only if $\boldsymbol{m}_j \le \boldsymbol{m}$. This follows from the fact that in the ith component, either $\boldsymbol{m}(i) = \boldsymbol{n}(i)$ or both $\boldsymbol{m}(i)$ and $\boldsymbol{n}(i)$ are at least as large as b, which in turn is at least as large as $\boldsymbol{m}_j(i)$. Therefore, the summations for

$(I^m)_\Delta$ and $(I^n)_\Delta$ involve exactly the same set of j, and, in fact differ only in that the first has I^{m-m_j} appearing in the place where the second has I^{n-m_j} appearing. Clearly $n \geq m$ so $I^{n-m_j} = I^{n-m}(I^{m-m_j})$. The second part of (a) follows from this.

For (b), suppose that K is a regular ideal in Δ and that $(I^n)_\Delta = (I^n K : K)$ for all $n \in \mathbb{N}_g$. Then $K(I^n)_\Delta \subseteq I^n K \subseteq I^n$, and so, $K\mathscr{R}_\Delta \subseteq \mathscr{R}$. Since K contains a regular element x of R (which remains regular in $\mathscr{R}$), we see that $\mathscr{R}_\Delta \subseteq \mathscr{R}x^{-1}$. Thus $\mathscr{R}$ is a finite $\mathscr{R}$-module, since $\mathscr{R}$ is Noetherian. □

2.2. Theorem. *Let $I_1, \dots, I_g$ be regular ideals. Assume that R is a locally analytically uramified ring with finite integral closure. Let Δ be any multiplicatively closed set of regular ideals such that $\{I^m \mid m \in \mathbb{N}_g\} \subseteq \Delta$. Then for $\mathscr{R}$ and $\mathscr{R}_\Delta$ as in Lemma 2.1:*

(a) *$\mathscr{R}_\Delta$ is a finite $\mathscr{R}$-module.*

(b) *There exists $K \in \Delta$, such that $(I^n)_\Delta = (I^n K : K)$ for all $n \in \mathbb{N}_g$.*

(c) *$\bigcup\{\operatorname{Ass} R/(I^n)_\Delta \mid n \in \mathbb{N}_g\}$ is a finite set.*

(d) *If $n_1 \leq n_2 \leq \cdots$ is an increasing sequence of elements from $\mathbb{N}_g$, then the sequence of sets $\operatorname{Ass} R(I^{n_1})_\Delta \subseteq \operatorname{Ass} R(I^{n_2})_\Delta \subseteq \cdots$ eventually stabilizes. In particular, there exists a $k \in \mathbb{N}_g$ such that $\operatorname{Ass} R(I^n)_\Delta$ is independent of n for all $n \geq k$.*

Proof. By [1, Lemma 1], $\mathscr{R}_\Delta$ is a finite $\mathscr{R}$-module. Thus (a) holds and (b) follows from Lemma 2.1. Part (d) follows from the proof of Theorem 1.7, once we prove (c). For this let $\mathscr{S} = \mathscr{R}[t_1^{-1}, \dots, t_g^{-1}]$ and $\mathscr{S}_\Delta = \mathscr{R}_\Delta[t_1^{-1}, \dots, t_g^{-1}]$. Then $\mathscr{S}_\Delta$ is a finite $\mathscr{S}$-module, and is therefore a Noetherian ring. Since $t^{-n}\mathscr{S}_\Delta \cap R = (I^n)_\Delta$ for all $n \in \mathbb{N}_g$, any $P \in \operatorname{Ass} R/(I^n)_\Delta$ lifts to an element of $\operatorname{Ass} \mathscr{S}_\Delta / t^{-n}\mathscr{S}_\Delta$. Since $\bigcup\{\operatorname{Ass} \mathscr{S}_\Delta / t^{-n}\mathscr{S}_\Delta \mid n \in \mathbb{N}_g\}$ is finite (as in the proof of Proposition 1.6), $\bigcup\{\operatorname{Ass} R/(I^n)_\Delta \mid n \in \mathbb{N}_g\}$ is finite, and the proof is complete. □

2.3. Corollary. *Let R be as above and $I_1, \dots, I_g$ regular ideals. Then there is an integer k such that for all $n \in \mathbb{N}_g$, $(I^n)^* = ((I^n)^{k+1} : (I^n)^k)$.*

Proof. We will find an integer $k(g)$ which satisfies the conclusion of the result for all $n \geq (1, \dots, 1)$. If n has some zero components, then we will delete those I_i for which $n(i) = 0$, and so will simply have a smaller value of g to deal with. Thus, the final k we take will be the maximum of the $k(d)$ over $1 \leq d \leq g$.

Assume $n \geq (1, \dots, 1)$. Then $(I^n)^* = (I^n)_\Delta$ by Lemma 1.2(b), assuming $\Delta = \{I^m \mid m \in \mathbb{N}_g\}$. By Theorem 2.2(b), there is an $I^c \in \Delta$ such that $(I^n)_\Delta = (I^{n+c} : I^c)$ for all $n \in \mathbb{N}_g$. Let $k(g)$ equal the maximum component of c. By Lemma 1.2, $(I^{n+c} : I^c) \subseteq ((I^n)^{k(g)+1} : (I^n)^{k(g)})$. Thus $(I^n)^* \subseteq ((I^n)^{k(g)+1} : (I^n)^{k(g)})$, and the reverse inclusion is by the definition of $(I^n)^*$. □

We close by mentioning two questions we have been unable to answer.

Question 1. If R is an arbitrary Noetherian ring and Δ a multiplicatively closed set of regular ideals containing $\{I^m \mid m \in \mathbb{N}_g\}$, do the sets $\operatorname{Ass} R/(I^n)_\Delta$ enjoy asymptotic stability? If this always holds for $g=1$ and $I_1=(b)$, b a regular element, then the answer is yes. In fact it is enough to know that $\bigcup\{\operatorname{Ass} R/(b^n)_\Delta \mid n \geq 1\}$ is finite.

Question 2. For which multiplicatively closed sets of ideals Δ does it hold that $\bigcup\{\operatorname{Ass} R/(I^m)_\Delta \mid m \in \mathbb{N}_g\} \subseteq \bigcup\{\operatorname{Ass} R/I^n \mid n \in \mathbb{N}_g\}$?

References

[1] D. Katz and L.J. Ratliff, Jr., On the symbolic Rees ring of a primary ideal, Comm. Algebra 14 (1986) 959–970.

[2] S. McAdam, Asymptotic Prime Divisors, Lecture Notes in Mathematics 1023 (Springer, Berlin, 1983).

[3] H. Matsumura, Commutative Algebra (Benjamin-Cummings, New York, 1980).

[4] L.J. Ratliff, Jr., Closure operations on ideals and rings, Trans. Amer. Math. Soc., to appear.

[5] L.J. Ratliff, Jr. and D. Rush, Two notes on reductions of ideals, Indiana Univ. Math. J. 27 (1978) 929–934.