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ON THE INTEGRAL CLOSURE OF RADICAL TOWERS IN MIXED CHARACTERISTIC

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We study the Cohen–Macaulay property of a particular class of radical extensions of an unramified regular local ring having mixed characteristic.

1. Introduction

In this note we consider the integral closure of certain radical towers in mixed characteristic $p > 0$. The motivation for our work is the following. Suppose S is an integrally closed Noetherian domain with quotient field L and $f \in S$ is square-free, i.e., $fS_Q = QS_Q$ for all height one primes $Q \subseteq S$ containing f. It is well-known that if the natural number $n \in S$ is a unit, then $R := S[\sqrt[n]{f}]$[1] is integrally closed[2]. If $f_1, \dots, f_r \in S$ are square-free and no two elements in the sequence are contained in the same height one prime of S, then $R := S[\sqrt[n]{f_1}, \dots, \sqrt[n]{f_r}]$ is integrally closed (see [1]). In each of these cases R is a free S-module, so that if S is a Cohen–Macaulay ring, then R is also Cohen–Macaulay. In other words, maintaining the same assumptions, the integral closure of S in the given radical extension of L is Cohen–Macaulay. When n is not a unit in S, the ring $S[\sqrt[n]{f_1}, \dots, \sqrt[n]{f_r}]$ need not be integrally closed. However, if S is regular, and R denotes the integral closure of S in $L(\sqrt[n]{f_1}, \dots, \sqrt[n]{f_r})$, it may be that R is Cohen–Macaulay, though it can be difficult to determine when R is Cohen–Macaulay. For example, if S is an unramified regular local ring of mixed characteristic $p > 0$ and $f \in S$ is square-free, then for R the integral closure of $S[\sqrt[p]{f}]$, R is Cohen–Macaulay (see [2, Lemma 3.2]). Unfortunately, this fails even when adjoining p-th roots of two square-free elements (see for example [6, Example 2.12] or [7, Example 4.8]). The difficulties in adjoining more than one p-th root stem from the behavior of the f_i modulo pS. However, as observed in [2; 6; 7] there is good behavior when the elements are p-th powers modulo p^2S. This is largely due to the existence of a unique unramified prime $P \subseteq R$ lying over pS such that the corresponding extension of residue fields is trivial. The purpose of this note is to generalize the results from [2; 6; 7] beyond the biradical case to what we call *class one radical towers*, i.e., radical

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[1]Throughout this paper, when we write $\sqrt[n]{f}$ for an element $f \in S$, we simply mean that f is a root of $X^n - f$, where X is an indeterminate over S.

[2]To see that $S[\sqrt[n]{f}]$ is integrally closed, set $g(X) = X^n - f$ and note that since n is a unit, the extension of quotient fields derived from $S \subseteq S[\sqrt[n]{f}]$ is separable, and thus $g'(\sqrt[n]{f})$, and hence nf, multiplies the integral closure of $S[\sqrt[n]{f}]$ into $S[\sqrt[n]{f}]$. Since n is a unit, one just has to check that $S[\sqrt[n]{f}]_P$ is a DVR for any height one prime $P \subseteq S[\sqrt[n]{f}]$ containing f (since $S[\sqrt[n]{f}]$ is free over S, and therefore satisfies Serre's condition S_2). But for such a prime, it is easy to see that $P_P = (\sqrt[n]{f})_P$ since f is square-free in S and thus $P_{P\cap S} = (f, \sqrt[n]{f})S_{P\cap S}[\sqrt[n]{f}]$.

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towers where the image of each f_i is a p-th power in S/p^2S. Note that to obtain good results for general p-th root towers, it already suffices to understand the case when the elements are p-th powers modulo pS (see Proposition 5.4). Here is the main result of this paper. Note that we allow S to be more general than an unramified regular local ring of mixed characteristic.

Theorem 1.1. *Let S be an integrally closed Noetherian domain and $p > 0$ a prime integer that is a nonzero prime in S. Suppose $f_1, \dots, f_r \in S$ are square-free and there exist $h_i \in S$ such that $f_i \equiv h_i^p \bmod p^2S$. Suppose further that no two f_i are contained in the same height one prime of S. Let $n_1, \dots, n_r > 1$ be integers such that $n_i = pd_i$ with each d_i a unit in S, and write R for the integral closure of $S[\sqrt[n_1]{f_1}, \dots, \sqrt[n_r]{f_r}]$. Then R is a free S-module. In particular, if S is Cohen–Macaulay or an unramified regular local ring, R is Cohen–Macaulay.*

The main focus of the proof of this theorem is when each $n_i = p$, so that we get a proper generalization of the results in [2; 6; 7]. We are able to reduce to this case by invoking the results that hold when the degree of the radical extension is a unit. In what follows, Section 2 deals with our conventions and preliminary results, while our main results are presented in Section 3. In Section 4, we present some applications of the main result in Section 3. In particular, in Theorem 4.1, we show that if R is the integral closure of an unramified regular local ring S of mixed characteristic in a particular type of radical tower that is not a class one tower, then R admits a small Cohen–Macaulay algebra, even if R itself is not Cohen–Macaulay.

2. Preliminaries

All rings considered in this paper are commutative and Noetherian. Throughout this paper S will denote an integrally closed Noetherian domain in which the prime $p \in \mathbb{Z}$ is a nonzero prime element in S. We will use L to denote the quotient field of S. Though our primary interest is in the case that S is an unramified regular local ring having mixed characteristic $p > 0$, our main results hold under more general conditions, so that the cases when S is an unramified regular local ring often appear as secondary statements. For the remainder of the paper, we will assume that the nonzero, nonunit elements $f_1, \dots, f_r \in S$ are square-free. We will also use the following notation throughout the rest of this paper.

Definitions 2.1. For S and $f_1, \dots, f_r \in S$ as above:

(i) The elements $f_1, \dots, f_r \in S$ are said to satisfy $\mathscr{A}_1$ if no two of these elements are contained in the same height one prime of S.

(ii) For positive integers a and b, we use $S^{a\wedge b}$ to denote the set of elements $f \in S$ whose images in S/bS are a-th powers. Note the following:

  (a) The set $S^{a\wedge b}$ is a multiplicatively closed subset of S for all positive integers a, b.
  (b) If $f \in S^{p\wedge p^2}$ is square-free, then $p \nmid f$.
  (c) The multiplicative subset $S^{p\wedge p}$ is a subring of S.

(iii) For square-free elements $f_1, \dots, f_r \in S$, we call $L(\sqrt[n_1]{f_1}, \dots, \sqrt[n_r]{f_r})$ a *a class one radical tower* if:

  (a) The set $\{f_1, \dots, f_r\}$ satisfies $\mathscr{A}_1$.
  (b) Each $f_i \in S^{p\wedge p^2}$.

(iv) For a commutative ring R of dimension at least one, we use the notation

$$\mathrm{NNL}_1(R) := \{P \in \mathrm{Spec}(R) \mid \mathrm{height}(P) = 1,\ R_P \text{ is not a DVR}\}.$$

(v) For a Noetherian ring R, a prime ideal $Q \subseteq R$ and $q_1, \ldots, q_r \in Q$, we use $(Q \mid q_1, \ldots, q_r)$ to indicate that $Q_Q = (q_1, \ldots, q_r)_Q$.

(vi) Throughout this paper, in any discussion involving $f_1, \ldots, f_r \in S$ and $n_1, \ldots, n_r \geq 2$, we set $A := S[\sqrt[n_1]{f_1}, \ldots, \sqrt[n_r]{f_r}]$, $K := L(\sqrt[n_1]{f_1}, \ldots, \sqrt[n_r]{f_r})$ and let R denote the integral closure of A, or equivalently, the integral closure of S in K.

For ease of reference, we include the following proposition, which can essentially be found as separate Propositions 5.2 and 5.3 in [1]. In these propositions from [1], it is assumed all roots have the same order. However, it is not difficult to see that the same proofs give rise to the following statement.

Proposition 2.2. *Let S be an integrally closed Noetherian domain and $n = n_1 \cdots n_r \in S$ a unit for some positive integer n. Let $f_1, \ldots, f_r \in S$ be square-free elements satisfying $\mathscr{A}_1$. Then*

(i) *$f_2, \ldots, f_r$ are square-free and satisfy $\mathscr{A}_1$ in $S[\sqrt[n_1]{f_1}]$.*

(ii) *$R = S[\sqrt[n_1]{f_1}, \ldots, \sqrt[n_r]{f_r}]$ is integrally closed.*

We make considerable use of the following observations.

Observations 2.3. (i) Let $f_1, \ldots, f_r \in S$ be square-free elements. Set $A := S[\sqrt[n_1]{f_1}, \ldots, \sqrt[n_r]{f_r}]$ and assume $\{n, f_1, \ldots, f_r\}$ satisfies $\mathscr{A}_1$, where $n = n_1 \cdots n_r$. Then $[K : L] = n$, where K denotes the quotient field of A. Moreover, $A[1/n]$ is integrally closed.

(ii) Let $f_1, \ldots, f_r \in S$ be square-free elements. Set $A := S[\sqrt[n_1]{f_1}, \ldots, \sqrt[n_r]{f_r}]$ and assume $\{n, f_1, \ldots, f_r\}$ satisfies $\mathscr{A}_1$, where $n = n_1 \cdots n_r$. Let Q denote the kernel of the natural homomorphism from $S[X_1, \ldots, X_r]$ to A, where the X_i are indeterminates over S. Then $Q = \langle X_1^{n_1} - f_1, \ldots, X_r^{n_r} - f_r \rangle$.

(iii) Let $S \subseteq C \subseteq D$ be extension of rings such that D is integral over S and is a domain. If C is regular in codimension one and D is birational to C, then D is regular in codimension one.

Proof. For (i), note that if $n \geq 1$ and T is an integrally closed domain of characteristic zero and $f \in T$ is square-free in T, then f is not a q-th power in the quotient field of T, for any prime q dividing n. Thus, by Theorem 9.1 in [5], $X^n - f$ is irreducible over the quotient field of T (equivalently over T), where X is an indeterminate over T. Otherwise, $X^n - f$ has a root in T, in which case f is not square-free in T. Now let K_i denote the quotient field of $S_i := S[\sqrt[n_1]{f_1}, \ldots, \sqrt[n_i]{f_i}]$. By what we have just shown, $[K_1 : L] = n_1$. Proceeding by induction on i, it suffices to show that $X_i^{n_i} - f_i$ is the minimal polynomial of $\sqrt[n_i]{f_i}$ over K_{i-1}, where X_i is an indeterminate over K_{i-1}. For this, by the $\mathscr{A}_1$ assumption, there is no harm in inverting n. But then, by Proposition 2.2, $S_{i-1}[1/n]$ is integrally closed and f_i is square-free in $S_{i-1}[1/n]$, and thus, $X_i^{n_i} - f_i$ is irreducible over K_i, which is what we want. It follows now that $[K : L] = n$. The second statement follows immediately from Proposition 2.2.

For (ii), we will use the following fact, which follows easily from the division algorithm. Suppose T is an integral domain with quotient field F, E a field extension of F and $\alpha \in E$ algebraic over F. Let $p(x)$ be the (monic) minimal polynomial of α over F and assume $p(x) \in T[x]$. Then $T[\alpha] \cong T[x]/\langle p(x) \rangle$. Now, proceeding by induction on r, if $r = 1$ we have that $S[\sqrt[n_1]{f_1}] \cong S[X_1]/\langle X_1^{n_1} - f_1 \rangle$, since, as in part (i), $X_1^{n_1} - f_1$ is the minimal polynomial for $\sqrt[n_1]{f_1}$ over L. From the proof of part (i), we also have that $X_r^{n_r} - f_r$ is the minimal polynomial of $\sqrt[n_r]{f_r}$ over the quotient field of $B := S[\sqrt[n_1]{f_1}, \dots, \sqrt[n_{r-1}]{f_{r-1}}]$. Thus, $B[\sqrt[n_r]{f_r}]$ is isomorphic to $B[X_r]/\langle X_r^{n_r} - f_r\rangle$. Since B is isomorphic to $S[X_1, \dots, X_{r-1}]/\langle X_1^{n_1} - f_1, \dots, X_{r-1}^{n_{r-1}} - f_{r-1}\rangle$, this gives us what we want.

For (iii), let P be a height one prime in D, $P_0 := C \cap P$ and $Q := S \cap P$. Since $S \subseteq D$ satisfies going down, Q has height one. Since C is integral over S, P_0 has height one. Thus, C_{P_0} is a DVR. Since C and D are birational, and $C_{P_0} \subseteq D_P$, we have $C_{P_0} = D_P$, so D is regular in codimension one. □

We include the following from [2] and [7] for easy reference:

Lemma 2.4 [2]. *Let $p \geq 3$ and write $p = 2k+1$. For $h \in S \setminus pS$ and W an indeterminate over S, if*

$$C := (W-h)^p - (W^p - h^p) = \sum_{j=1}^{k} (-1)^{j+1} \binom{p}{j} (W \cdot h)^j [W^{p-2j} - h^{p-2j}], \tag{2.4.1}$$

$C' := C \cdot (p(W-h))^{-1}$ and $\widetilde{P} := (p, W-h)S[W]$, then $C' \notin \widetilde{P}$.

Lemma 2.5 [7]. *Let $p \geq 3$ and write $p = 2k+1$. For $h \in S \setminus pS$ and W an indeterminate over S, suppose C' is as defined in Lemma 2.4. Then $C' \equiv h^{p-1}$ mod $(p, W-h)S[W]$.*

Our work in Section 3 relies heavily on Lemma 3.2 in [2]. Let $f \in S$ be square-free such that $f \in S^{p \wedge p^2}$, say $f = h^p + p^2 g$. Take ω satisfying $\omega^p = f$, and set

$$\tau := \frac{\omega^{p-1} + \cdots + h^{p-1}}{p} = \frac{pg}{\omega - h}. \tag{2.5.1}$$

Then [2, Lemma 3.2] shows that R, the integral closure of $S[\omega]$, equals $S[\omega, \tau]$ and that R is a free S-module. The proof relies on showing that $Q_1 := (p, \omega - h, \tau)$ and $Q_2 := (p, \omega - h, \tau - c')$ are the only height two primes in $S[\omega, \tau]$ containing p, where c' is the image in $S[\omega]$ of the element C' in Lemma 2.4 under the map sending $W \to \omega$. This is done by noting that τ satisfies $l(T) := T^2 - c'T - g(\omega - h)^{p-2}$ when $p > 2$ and that if $\widetilde{Q}$ is a height two prime in $S[\omega, T]$ containing $l(T)$, then

$$\widetilde{Q_1} := (p, \omega - h, T)S[\omega, T] \quad \text{or} \quad \widetilde{Q_2} := (p, \omega - h, T - c')S[\omega, T].$$

However, the proof neglected to show that the kernel of the natural map from $S[\omega, T]$ to $S[\omega, \tau]$ is contained in these primes. The next lemma closes this small gap.

Lemma 2.6. *Let $f \in S^{p \wedge p^2}$ be square-free, $f = h^p + p^2 g$ and $\omega^p = f$ a p-th root in a field extension of L. Define τ as in (2.5.1). Consider the natural map $\phi : S[\omega, T] \to S[\omega, \tau]$, where T is an indeterminate over $S[\omega]$. Then $\operatorname{Ker}(\phi)$ is generated by*

$$l(T) := T^2 - c'T - g(\omega - h)^{p-2}, \quad m(T) := pT - (\omega^{p-1} + \cdots + h^{p-1}), \quad n(T) := (\omega - h)T - pg$$

if $p > 2$ and $l_0(T) = T^2 - hT - g, m(T), n(T)$ if $p = 2$. In particular, for $\widetilde{Q_1}$ and $\widetilde{Q_2}$ in the paragraph above, $\widetilde{Q_1}$ and $\widetilde{Q_2}$ contain $\operatorname{Ker}(\phi)$.

Proof. Suppose $p > 2$. The proof of [2, Lemma 3.2] shows that $l(T) \in \operatorname{Ker}(\phi)$. Thus, we need only find linear generators in $\operatorname{Ker}(\phi)$. Suppose $a(\omega)T - b(\omega) \in \operatorname{Ker}(\phi)$. By definition,

$$a(\omega)(\omega^{p-1} + \cdots + h^{p-1}) - b(\omega)p = 0$$

in $S[\omega]$. Noting that $S[\omega] = S[W]/\langle W^p - f\rangle$, it follows that

$$a(W)(W^{p-1} + \cdots + h^{p-1}) - b(W)p = q(W)(W^p - f),$$

for some $q(W) \in S[W]$. Therefore,

$$a(W)(W^{p-1} + \cdots + h^{p-1}) - b(W)p - q(W)(W^p - h^p - p^2 g) = 0$$

and thus

$$\{a(W) - q(W)(W - h)\}(W^{p-1} + \cdots + h^{p-1}) + \{-b(W) + pgq(W)\}p = 0.$$

Since $W^{p-1} + \cdots + h^{p-1}$, p form a regular sequence in $S[W]$, we have

$$a(W) = \lambda(W)(-p) + q(W)(W - h) \quad \text{and} \quad -b(W) = \lambda(W)(W^{p-1} + \cdots + h^{p-1}) + q(W)(-pg)$$

for some $\lambda(W) \in S[W]$. It follows that in $S[\omega, T]$,

$$a(\omega)T - b(\omega) = -\lambda(\omega) \cdot m(T) + q(\omega) \cdot n(T),$$

which is what we want. If $p = 2$, the proof is similar (though, in fact, easier), as [2, Lemma 3.2] shows that $l_0(T)$ belongs to $\mathrm{Ker}(\phi)$. For the second statement, $l(T)$ and $n(T)$ are clearly contained in each $\widetilde{Q}_i$. That $m(T)$ is contained in each $\widetilde{Q}_i$ follows from the fact that we can write $\omega^{p-1} + \cdots + h^{p-1}$ as $(\omega - h) \cdot g(\omega) + ph^{p-1}$, for some $g(\omega) \in S$. □

3. Class one radical towers

In this section we present our main result. The crucial case for this result is Theorem 3.4 below, which deals with the case of adjoining p-th roots of square-free elements. Before addressing Theorem 3.4, we need a few preliminary results.

Lemma 3.1. *Let $f_1, \dots, f_r \in S$ be such that $K := L(\sqrt[p]{f_1}, \dots, \sqrt[p]{f_r})$ is a class one radical extension of L. Set $\omega_i := \sqrt[p]{f_i}$. Let R_i denote the integral closure of $S[\omega_i]$ and write V for the join of $R_1, \dots, R_r$. Recalling from the previous section that $R_i = S[\omega_i, \tau_i]$, for τ_i as in Lemma 2.6, let W_i, T_i be indeterminates over S and let $\phi_i : S[W_i, T_i] \to R_i$ be the natural surjection. Write A_i for the kernel of ϕ_i. Then V is a free S-module and the kernel of the natural homomorphism from $S[W_1, T_1, \dots, W_r, T_r]$ to V is generated by $A_1 + \cdots + A_r$.*

Proof. We let K_i denote the quotient field of $S[\omega_i]$, so that $K = K_1 \cdots K_r$ and $K_1 \cdots K_r$ is the compositum of the K_i. We induct on r. Suppose $r = 2$. Since $[K : L] = p^2$ by Observations 2.3(i), K_1 and K_2 are linearly disjoint over L and a basis for K over L is obtained by taking the products of the basis elements for each K_i over L. Since each R_i is free over S, we may take bases from each R_i to serve as the bases for K_i. Thus, the product of the bases for the R_i are linearly independent over S. Since these products span V, we have that V is a free S-module. Thus, the canonical map $R_1 \otimes_S R_2 \to V$ is a surjection of finite free S-modules of the same rank and hence an isomorphism. Moreover, this map is a ring homomorphism, so that it is an isomorphism of S-algebras. Thus the natural map from $(S[W_1, T_1]/A_1) \otimes_S (S[W_2, T_2]/A_2) \to V$ is an isomorphism. Since $S[W_1, T_1]/A_1 \otimes_S S[W_2, T_2]/A_2 \cong S[W_1, T_1, W_2, T_2]/\langle A_1 + A_2\rangle$, this gives us what we want.

The proof of the inductive step is essentially the same as the case $r = 2$. If $r > 2$, set $\widetilde{K} = K_1 \cdots K_{r-1}$ and let $\widetilde{V}$ denote the join of $R_1, \ldots, R_{r-1}$, so that $[\widetilde{K} : L] = p^{r-1}$ by Observations 2.3(i). By induction, $\widetilde{V}$ is a free S-module, and the kernel of the natural map from $S[W_1, T_2, \ldots, W_{r-1}, T_{r-1}]$ to $\widetilde{V}$ is generated by $A_1 + \cdots + A_{r-1}$. Since $K = \widetilde{K} \cdot K_r$ and $[K : L] = [\widetilde{K} : L] \cdot [K_r : L]$, $\widetilde{K}$ and K_r are linearly disjoint over L. Now we can repeat the argument in the paragraph above on $\widetilde{V}$ and R_r to complete the proof, since V is the join of $\widetilde{V}$ and R_r. $\square$

Lemma 3.2. *Let S be an integrally closed domain, and $p \geq 3$ a prime number that remains prime in S. Let $f_1, \ldots, f_r$ be square-free elements in S and $\omega_i^p = f_i$ be p-th roots in some field extension of L. Suppose that $K := L(\omega_1, \ldots, \omega_r)$ is a class one radical tower. Choose $h_i \in S$ such that $f_i \equiv h_i^p \bmod p^2S$. Let R_i denote the integral closure of $S[\omega_i]$ for $1 \leq i \leq r$ and let V be the join of the R_i. The following hold*:

(i) *There are 2^r height one primes in V containing p and at most $2^r - r - 1$ of them are singular.*

(ii) *If $Q \subseteq V$ is a height one prime containing p, then the nonsingular primes are either of the form $(Q \mid p)$ or $(Q \mid \omega_i - h_i)$. The (possibly) singular ones are of the form $Q_{(i_1,\ldots,i_l)} := (Q \mid \omega_{i_1} - h_{i_1}, \ldots, \omega_{i_l} - h_{i_l})$ for some $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, r\}$, $i_1 < \cdots < i_l$ and $l \geq 2$.*

Proof. We retain the notation from the previous lemma and its proof. First note that there are 2^r height one primes in V containing p. From [2, Lemma 3.2], we have that $Q_{i,1} := (p, \omega_i - h_i, \tau_i)R_i$ and $Q_{i,2} := (p, \omega_i - h_i, \tau_i - c_i')R_i$ are the only height one primes in R_i lying over pS. Moreover, from the proof of [2, Lemma 3.2], for each $1 \leq i \leq r$, we have

$$(*) \qquad Q_{i,1} = (Q_{i,1} \mid \omega_i - h_i) \quad \text{and} \quad Q_{i,2} = (Q_{i,2} \mid p).$$

Let $\widetilde{Q_{i,j}}$ denote the preimage of $Q_{i,j}$ in $S[W_i, T_i]$ under the natural map. From Lemma 2.6, $\widetilde{Q_{i,j}} = (p, W_i - h_i, T_i - q)$ for some $q \in S[W_i]$. It then follows from Lemma 3.2 that the ideal generated by $\sum_{i=1}^n \widetilde{Q_{i,j_i}}$ contains the kernel of the natural map from $S[W_1, T_1, \ldots, W_r, T_r]$ to V for any choice of $j_i \in \{1, 2\}$. Any such ideal is clearly prime of height $2n + 1$ and hence their images in V account for 2^r distinct height one primes containing p. On the other hand, if Q_0 is a height one prime in V containing p, then, for each $1 \leq i \leq r$, $Q_0 \cap R_i = Q_{i,j_i}$ for some $j_i \in \{1, 2\}$, since $Q_0 \cap R_i$ must be a height one prime containing p (R_i is integrally closed). Thus Q_0 contains the ideal generated by $\sum_{i=1}^n Q_{i,j_i}$ and thus Q_0 is one of the 2^r height one primes we have accounted for. Therefore, there are exactly 2^r height one primes in V containing p.

Now, let $Q \subset V$ be a height one prime containing p. If $Q = \langle Q_{1,2} + \cdots + Q_{r,2} \rangle$, then it follows from $(*)$ that $Q_Q = pV_Q$. If there is a single i for which $\tau_i - c_i' \notin Q$, then it follows from $(*)$ that $\omega_i - h_i$ is a local generator for Q. Thus, there are at most $2^r - r - 1$ singularities in codimension one in V. Finally, if $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, r\}$ is such that $i_1 < \cdots < i_l$, $l \geq 2$ and $\tau_{i_j} - c_{i_j}' \notin Q$ for all $1 \leq j \leq l$, then it follows that Q is of the form $(Q \mid \omega_{i_1} - h_{i_1}, \ldots, \omega_{i_l} - h_{i_l})$. $\square$

Remark 3.3. Although the standing assumption in [2] requires S/pS to be integrally closed, note that the conclusion and proof of [2, Lemma 3.2] holds without this additional hypothesis. In particular, if $f \in S^{p \wedge p^2}$ is square-free, the conclusion and proof of [loc. cit.] holds without the hypothesis that S/pS be integrally closed.

The following theorem is the crucial case for the main result of this paper. It extends the corresponding results from [2; 6; 7].

Theorem 3.4. *Let S be an integrally closed domain and p a prime integer that remains a nonzero prime in S. Let $f_1, \dots, f_r$ be square-free elements in S and $\omega_i^p = f_i$ be p-th roots in some field extension of L. Suppose that $K := L(\omega_1, \dots, \omega_r)$ is a class one radical tower. Then R, the integral closure of S in K, is a free S-module. In particular, if S is Cohen–Macaulay or an unramified regular local ring, then R is Cohen–Macaulay.*

Proof. The second statement follows immediately from the first, so we only concern ourselves with the general case. We first assume $p \geq 3$. Note that R is the integral closure of the ring $S[\omega_1, \dots, \omega_r]$. Let R_i be the integral closure of $S[\omega_i]$ and let $V \subseteq R$ denote the join of the R_i. Note that V is S-free of rank p^r by Lemma 3.1. For each $1 \leq i \leq r$, choose h_i so that $f_i - h_i^p \in p^2 S$. From Lemma 3.2, we have the following data:

(a) There are exactly 2^r height one primes in V containing p.

(b) If $Q \subseteq V$ is a height one prime containing p, then the nonsingular primes are either of the form $(Q \mid p)$ or $(Q \mid \omega_i - h_i)$.

(c) The (possibly) singular ones are of the form $Q_{(i_1,\dots,i_l)} := (Q \mid \omega_{i_1} - h_{i_1}, \dots, \omega_{i_l} - h_{i_l})$ for some $\{i_1, \dots, i_l\} \subseteq \{1, \dots, r\}$, $i_1 < \cdots < i_l$ and $l \geq 2$.

(d) There are at most $2^r - r - 1$ singularities in codimension one in V.

Note that from Observations 2.3(iii), V is regular in codimension one outside of the $Q_{(i_1,\dots,i_l)}$. We now identify an R_1-ification of V. For $i, j \in \{1, \dots, n\}$, $i < j$, define

$$\eta_{ij} := p^{-1}(\omega_i - h_i)^{p-2}(\omega_j - h_j) \in K.$$

Then for X an indeterminate over V, η_{ij} satisfies the integral equation

$$\upsilon_{ij}(X) := X^{p-1} - (\tau_i - c_i')^{p-2}(\tau_j - c_j') \in V[X].$$

To desingularize $Q_{(i_1,\dots,i_l)}$ consider the finite birational extension $V \hookrightarrow V_{(i_1,\dots,i_l)} := V[\eta_{i_1 i_2}, \dots, \eta_{i_1 i_l}]$. From Lemma 2.5 we have[3]

$$\begin{aligned} \upsilon_{i_1 i_2}(X) &\equiv X^{p-1} - (h_{i_1}^{p-2} h_{i_2})^{p-1} \quad \bmod (Q_{(i_1,\dots,i_l)} V[X]) \\ &\equiv \prod_{k=1}^{p-1} (X + k h_{i_1}^{p-2} h_{i_2}) \quad \bmod (Q_{(i_1,\dots,i_l)} V[X]). \end{aligned}$$

Since $S_{(p)}$ is universally catenary, it follows that height one primes in $V[\eta_{i_1 i_2}]$ lying over $Q_{(i_1,\dots,i_l)}$ are of the form

$$Q_{[(i_1,\hat{i_2},\dots,i_l)|k]} := (Q_{(i_1,\dots,i_l)}, \eta_{i_1 i_2} + k h_{i_1}^{p-2} h_{i_2})$$

for $1 \leq k \leq p-1$. The point is that $Q_{[(i_1,\hat{i_2},\dots,i_l)|k]}$ locally has one less generator: it is of the form

$$(Q_{[(i_1,\tilde{i_2},\dots,i_l)|k]} \mid (\omega_{i_1} - h_{i_1}, \widehat{\omega_{i_2} - h}_{i_2}, \dots, \omega_{i_l} - h_{i_l})).$$

[3]If R is a ring of characteristic p and X, Y indeterminates over R, $X^{p-1} - Y^{p-1} = \prod_{i=1}^{p-1}(X + iY)$.

To see this, note that $\prod_{i=1,i\neq k}^{p-1}(\eta_{i_1i_2}+ih_{i_1}^{p-2}h_{i_2})\notin Q_{[(i_1,\hat{i_2},\ldots,i_l)||k]}$, so that $\eta_{i_1i_2}+kh_{i_1}^{p-2}h_{i_2}$ is locally a redundant generator. Also, since $(\omega_{i_1}-h_{i_1})\cdot\eta_{i_1i_2}=(\omega_{i_2}-h_{i_2})(\tau_{i_1}-c'_{i_1})$ and $(\tau_{i_1}-c'_{i_1})\notin Q_{[(i_1,\hat{i_2},\ldots,i_l)||k]}$, $\omega_{i_2}-h_{i_2}$ is a redundant generator locally.

Proceeding inductively, it is clear that in $V_{(i_1,\ldots,i_l)}$ all height one primes lying over $Q_{(i_1,\ldots,i_l)}$ (at most $(p-1)^{l-1}$) are nonsingular. Now set

$$\mathscr{R}V:=V\big[\{p^{-1}(\omega_{i_1}-h_{i_1})^{p-2}(\omega_{i_2}-h_{i_2})\}_{\{i_1,i_2\in\{1,\ldots,r\}\,|\,i_1<i_2\}}\big].$$

For all possible $\{i_1,\ldots,i_l\}\subseteq\{1,\ldots,r\}$, $i_1<\cdots<i_l$ and $l\geq 2$, $V\hookrightarrow V_{(i_1,\ldots,i_l)}\hookrightarrow\mathscr{R}V$ are finite birational extensions. From Observations 2.3(iii), $\mathscr{R}V$ is an R_1-ification for V.

We now identify a finite birational overring of $\mathscr{R}V$ that is S-free. This ring would then inherit R_1 from $\mathscr{R}V$ by Observations 2.3(iii) and the proof would be complete, since a free S-module also satisfies Serre's condition S_2. The rest of the proof concerns identifying this overring.

Note that $A=S[\omega_1,\ldots,\omega_r]$ is S-free of rank p^r with a basis given by

$$F:=\left\{\prod_{i=1}^{n}(\omega_i-h_i)^{j_i}\;\middle|\;0\leq j_i\leq p-1\right\}.$$

For each $1\leq i\leq r$, define $\Gamma_i:F\to\mathbb{N}\cup\{0\}$ by $\Gamma_i((\omega_1-h_1)^{j_1}\cdots(\omega_i-h_i)^{j_i}\cdots(\omega_r-h_r)^{j_r})=j_i$ and

$$\Gamma:F\to(\mathbb{N}\cup\{0\})^r,\quad f\mapsto(\Gamma_1(f),\ldots,\Gamma_r(f)).$$

Let $\Omega:(\mathbb{N}\cup\{0\})^r\to\mathbb{N}\cup\{0\}$ be the map sending $(x_1,\ldots,x_r)\mapsto x_1+\cdots+x_r$. For every $0\leq k\leq r$, set

$$\mathscr{V}_k:=\big\{p^{-k}\cdot m\mid m\in(\Omega\Gamma)^{-1}([(p-1)k,(p-1)(k+1)))\big\}$$

and $\mathscr{V}:=\bigcup_{0\leq k\leq r}\mathscr{V}_k$. By definition, the sets $\mathscr{V}$ and F are in bijection and it follows that $\langle\mathscr{V}\rangle_S$ is S-free of rank p^r. Note that $A\subseteq\langle\mathscr{V}\rangle_S$ and the ring generators of $\mathscr{R}V$ over A are all in $\mathscr{V}$. Suppose that $\langle\mathscr{V}\rangle_S$ is an S-algebra. Then $\mathscr{R}V\hookrightarrow\langle\mathscr{V}\rangle_S$ would be a birational module finite map of rings, so that by Observations 2.3(iii), $R=\langle\mathscr{V}\rangle_S$.

Thus, it only remains to be shown that $\langle\mathscr{V}\rangle_S$ is an S-algebra. We first note that it is an A-module. To see this, it suffices to show that multiplication by ring generators of A over S define an endomorphism on $\langle\mathscr{V}\rangle_S$. Set $E_i:=\big\langle\bigcup_{0\leq k\leq i}\mathscr{V}_k\big\rangle_A$ for each $0\leq i\leq r$. We proceed by induction on i. Clearly, $E_0\subseteq E$. Now assume that $E_{i-1}\subseteq E$ for some $1\leq i\leq r$. Consider $p^{-i}\cdot m\in\mathscr{V}_i$ and ω_j-h_j for some $1\leq j\leq r$. If $\Gamma_j(m)\neq p-1$, then clearly $(\omega_j-h_j)\cdot p^{-i}\cdot m\in\langle\mathscr{V}\rangle_S$. Suppose that $\Gamma_j(m)=p-1$. Let c'_j denote the image in $S[\omega_j]$ of the element C' in Lemma 2.4 under the map sending $W\to\omega_j$, where we take $h=h_j$. From the relation (see Lemma 2.4)

$$(\omega_j-h_j)^p=\omega_j^p-h_j^p+c'_jp(\omega_j-h_j)\equiv c'_jp(\omega_j-h_j)\mod(p^2A),\tag{3.4.1}$$

we see that $(\omega_j-h_j)\cdot p^{-i}\cdot m\in E_{i-1}\subseteq\langle\mathscr{V}\rangle_S$. Thus $E_i\subseteq\langle\mathscr{V}\rangle_S$ and by induction, $\langle\mathscr{V}\rangle_S$ is an A-module. To finish the proof, it suffices to show that the product of two elements from $\mathscr{V}$ lies in $\langle\mathscr{V}\rangle_S$. This is verified below. Consider $x:=p^{-u}m_u\in\mathscr{V}_u$ and $y:=p^{-v}m_v\in\mathscr{V}_v$. If $m_u\cdot m_v\in F$ then $xy\in\mathscr{V}$ and we are done. Suppose $m_u\cdot m_v\notin F$. Write

$$m_um_v=\alpha\cdot\prod_{j=l+1}^{r}(\omega_{i_j}-h_{i_j})^{a_j},$$

where $0 \leq l \leq r-1$ is such that $a_j \geq p$ for all $l+1 \leq j \leq r$ and $\alpha \in F$. For each $l+1 \leq j \leq r$, write using (3.4.1)

$$(\omega_{i_j} - h_{i_j})^p = p^2 b_{i_j} - pc'_{i_j}(\omega_{i_j} - h_{i_j})$$

for some $b_{i_j} \in S$. Setting $\beta := \alpha \prod_{j=l+1}^{r} (\omega_{i_j} - h_{i_j})^{a_j - p}$, we have that $\beta \in F$ and $\Gamma_j(\beta) \leq p-2$ for all $l+1 \leq j \leq r$. We thus have

$$xy = p^{-(u+v-r+l)} \beta \prod_{j=l+1}^{r} (pb_{i_j} - c'_{i_j}(\omega_{i_j} - h_{i_j})).$$

We verify that every monomial in the above expression lies in $\langle \mathscr{V} \rangle_S$. First consider

$$p^{-(u+v-r+l)} \beta \prod_{j=l+1}^{r} (pb_{i_j}) \in S \cdot p^{-(u+v-2r+2l)} \beta.$$

We have

$$(p-1)u + (p-1)v - (2p-2)(r-l) \leq (p-1)u + (p-1)v - p(r-l) \leq \Omega\Gamma(\beta)$$

so this implies $S \cdot p^{-(u+v-2r+2l)} \beta \subseteq \langle \mathscr{V} \rangle_S$. Next, consider the monomial

$$p^{-(u+v-r+l)} \beta \prod_{j=l+1}^{r} (-1) c'_{i_j}(\omega_{i_j} - h_{i_j}).$$

Now $\beta' := \beta \prod_{j=l+1}^{r} (\omega_{i_j} - h_{i_j}) \in F$ since $\Gamma_j(\beta) \leq p-2$ for all $l+1 \leq j \leq r$. Since $\langle \mathscr{V} \rangle_S$ is an A-module, it suffices to show $p^{-(u+v-r+l)} \beta' \in \langle \mathscr{V} \rangle_S$. But this is the case since

$$(p-1)(u+v-r+l) \leq \Omega\Gamma(m_u) + \Omega\Gamma(m_v) - (p-1)(r-l) = \Omega\Gamma(\beta').$$

Finally, for the case of a general monomial, it suffices to show by symmetry that for each $l+1 \leq t \leq r-1$,

$$p^{-(u+v-r+l)} \beta \biggl(\prod_{j=l+1}^{t} pb_{i_j} \biggr) \biggl(\prod_{j=t+1}^{r} (-1) c'_{i_j}(\omega_{i_j} - h_{i_j}) \biggr) \in \langle \mathscr{V} \rangle_S.$$

Since $\langle \mathscr{V} \rangle_S$ is an A-module, it suffices to show that $p^{-(u+v+2l-r-t)} \beta \prod_{j=t+1}^{r} (\omega_{i_j} - h_{i_j}) \in \langle \mathscr{V} \rangle_S$. Note that $\beta' = \beta \prod_{j=t+1}^{r} (\omega_{i_j} - h_{i_j}) \in F$. We have

$$\begin{aligned}
\Omega\Gamma(\beta') &= \Omega\Gamma(m_u) + \Omega\Gamma(m_v) - p(r-l) + (r-t) \\
&\geq (p-1)u + (p-1)v - p(r-l) + (r-t) \\
&= (p-1)(u+v-r) + pl - t \\
&\geq (p-1)(u+v-r) + pl - t + (p-2)(l-t) \\
&= (p-1)(u+v-r) + (2p-2)l - (p-1)t,
\end{aligned}$$

so that $p^{-(u+v+2l-r-t)} \beta' \in \langle \mathscr{V} \rangle_S$ and thus the proof is complete.

Now suppose that $p = 2$. From the proof of [2, Lemma 3.2] we know that the integral closure of $S[\omega_i]$ is $R_i = S[\tau_i]$ for $\tau_i := \frac{1}{2} \cdot (\omega_i + h_i)$. It also tells us that if $f_i = h_i^p + 4g_i$, then τ_i satisfies

$l_i(T) := T^2 - h_i T - g_i \in S[T]$, where T is an indeterminate over S. Since $l_i(T)$ and $l_i'(T)$ are relatively prime over the quotient field of $S/2S$, $2 \in S[\tau_i]$ is square-free. Finally, we also know that R_i is S-free for all i. Let V denote the join of the R_i, that is, $V = S[\tau_1, \dots, \tau_r]$. By Lemma 3.1, V is a free S-module. Note that V is birational to R and satisfies Serre's condition S_2. From Observations 2.3(i), we have that $V[1/2]$ is integrally closed. Moreover, $2 \in V$ is square-free since for each $2 \le i \le r$, $l_i(T)$ and $l_i'(T)$ are relatively prime over the quotient field of $S[\tau_1, \dots, \tau_{i-1}]/Q$ for all height one primes $Q \subseteq S[\tau_1, \dots, \tau_{i-1}]$ containing 2. Thus V is regular in codimension one and hence $V = R$ is a free S-module. □

Here is the main theorem of this paper.

Theorem 3.5. *Let S be an integrally closed domain with fraction field L and p a prime integer that remains a nonzero prime in S. Let $f_1, \dots, f_r$ be square-free elements such that $K := L(\sqrt[n_1]{f_1}, \dots, \sqrt[n_r]{f_r})$ is a class one radical tower, where, for each $1 \le i \le r$, $n_i = pd_i$ and $d_i \in S$ is a unit. Then the integral closure of S in K is a free S-module. If S is Cohen–Macaulay or an unramified regular local ring, then the integral closure of S in K is Cohen–Macaulay.*

Proof. Write $n_i = pd_i$, so that d_i is a unit in S, for all $1 \le i \le r$. Note that this condition is the same as assuming $p \mid n_i$ but $p^2 \nmid n_i$, if S were local. Set $\omega_i := (\sqrt[n_i]{f_i})^{d_i}$, so that ω_i is a p-th root of the square-free element f_i. Let R_0 denote the integral closure of S in $K_0 := L(\omega_1, \dots, \omega_r)$, a class one radical extension of L. Then, by Theorem 3.4, R_0 is free over S. We now claim that each ω_i is square-free in R_0. Set $A_0 := S[\omega_1, \dots, \omega_r]$. Since p does not divide any f_i, if $Q \subseteq R_0$ is a height one prime containing ω_i, then $p \notin Q$. Thus, $(R_0)_Q = (R_0[1/p])_Q = (A_0)_{Q \cap A_0}$. Thus, it suffices to show that each ω_i is square-free in A_0. Without loss of generality, we may assume $i = 1$.

Let Q be a height one prime in A_0 containing ω_1. Since $p \notin Q$, it suffices to show that ω_1 is square-free in $B[\omega_1]$, where $B := S[1/p, \omega_2, \dots, \omega_r]$. By Proposition 2.2, B is integrally closed and f_1 is square-free in B. By Observations 2.3(ii), $B[\omega_1] \cong B[X]/\langle X^p - f_1\rangle$. Thus, Q corresponds to a height two prime Q' in $B[X]$ containing X and $X^p - f_1$, and thus, Q' is a height two prime containing X and f_1. It follows that $Q' = \langle X, P\rangle$, where $P \subseteq B$ is a height one prime containing f_1. Since $PB_P = f_1 B_P$, we have $Q'B[X]_{Q'} = (X, f_1)B[X]_{Q'}$. Therefore, modding out $X^p - f_1$ gives $QB[\omega_1]_Q = \omega_1 B[\omega_1]_Q$, which is what we want.

Now, we may regard each $\sqrt[n_i]{f_i}$ as a d_i-th root of ω_i, and we write $\sqrt[d_i]{\omega_i} = \sqrt[n_i]{f_i}$. If we let K_0 denote the quotient field of R_0 and R denote the integral closure of S in K, then R is the integral closure of R_0 in $K_0(\sqrt[d_1]{\omega_1}, \dots, \sqrt[d_r]{\omega_r})$. Each d_i is a unit in R_0 and by the paragraph above, ω_i is square-free in R_0. Moreover, the going down property in the extension $S \subseteq R_0$ ensures that the set $\{\omega_1, \dots, \omega_r\}$ satisfies $\mathscr{A}_1$ in R_0. Thus, by Proposition 2.2, $R = R_0[\sqrt[d_1]{\omega_1}, \dots, \sqrt[d_r]{\omega_r}]$. It follows that R is a free R_0-module. Since R_0 is a free S-module, we have that R is a free S-module, which is what we want. The second statement follows immediately from the first. □

4. Applications

In this section we focus our attention on applications of the results from the previous section to the case that S is an unramified regular local ring of mixed characteristic $p > 0$. Our first applications of Theorem 3.5 deal with cases where we adjoin n-th roots of elements in S such that the resulting extension need not be a class one radical extension. The resulting integral closure R may not be Cohen–Macaulay, but does have a small Cohen–Macaulay algebra, i.e., a module finite R-algebra $\widetilde{R}$ such that $\widetilde{R}$ is Cohen–Macaulay.

For the theorem below, we assume S to be an unramified regular local ring of mixed characteristic p. Let $\{p, x_2, \dots, x_d\} = \{p, \underline{x}\}$ be a minimal generating set for the maximal ideal of S. For all $k \geq 1$, we set $T_k(\underline{x}) := S[\sqrt[k]{x_2}, \dots, \sqrt[k]{x_d}]$. We choose k-th roots of the x_i so that if $k = ab$, then $\left(\sqrt[k]{x_i}\right)^a = \sqrt[b]{x_i}$, for all i. Thus, if $k \mid l$, then $T_k(\underline{x}) \subseteq T_l(\underline{x})$. It follows from Observations 2.3(ii) that $T_k(\underline{x})$ is an unramified regular local ring of mixed characteristic p, for all $k \geq 1$. Set $W(\underline{x}) := \bigcup_{k \geq 1}(T_k(\underline{x})^{p \wedge p^2} \cap S)$. It is not difficult to see that if we take the union of $W(\underline{x})$, as $\{\underline{x}\}$ ranges over regular systems of parameters for S, this set is considerably larger than $S^{p \wedge p^2}$. For example, if $f = m + h^p$, where m is a monomial in $x_2, \dots, x_d$, which are part of a regular system of parameter for S, then $f \in W(\underline{x})$.

Theorem 4.1. *Let S be an unramified regular local ring of mixed characteristic p. Assume $f_1, \dots, f_r$ are square-free, satisfy $\mathscr{A}_1$ and belong to $W(\underline{x})$, for $\underline{x} = x_2, \dots, x_d$ such that $p, x_2, \dots, x_d$ is a regular system of parameters. Let $\omega_i^{n_i} = f_i$ be roots in some field extension of L such that $p \mid n_i$ and $p^2 \nmid n_i$ for each i, and write R for the integral closure of S in $L(\omega_1, \dots, \omega_r)$. Then R admits a small Cohen–Macaulay algebra.*

Proof. Set $T_k = T_k(\underline{x})$, for all k. The strategy is to replace S by T_k for some k and apply Theorem 3.4. Because we wish to pass to T_k for some k, f_j will not be square-free in T_k if f_j is divisible by some x_i. So we first need to remove any factor of x_i appearing in any f_j. After re-indexing, we may assume that $x_2, \dots, x_t$ are exactly those x_i appearing as factors among the f_j. Because the elements $f_1, \dots, f_r$ satisfy $\mathscr{A}_1$, for any x_i with $2 \leq i \leq t$, there is exactly one f_j such that x_i divides f_j. We may re-index the f_j to assume x_i divides f_{i-1}, for $2 \leq i \leq t$, and therefore no x_i divides any f_j if $i > t$. For $2 \leq i \leq t$, write $f_{i-1} = x_i f'_{i-1}$.

Now, by assumption, each f_j is in $T_{k_j}^{p \wedge p^2}$ for some k_j, with $1 \leq j \leq r$. If we take $k := k_1 \cdots k_r \cdot p$, then we have $f_j \in T_k^{p \wedge p^2}$, for all j, and moreover, $p \mid k$. We can do this since if $a \mid b$, then $T_a^{p \wedge p^2} \subseteq T_b^{p \wedge p^2}$.

We start by removing the factor x_2 from f_1. We wish to prove that $f'_1 \in T_k^{p \wedge p^2}$. Now, by assumption, in T_k we have $f_1 = h^p + bp^2$. Thus,

$$(x_2 f'_1) = (\sqrt[p]{x_2})^p f'_1 = h^p + bp^2,$$

so that in T_k/pT_k, we have $(\sqrt[p]{x_2})^p f'_1 \equiv h^p$. Since T_k/pT_k is a UFD, it follows that in T_k/pT_k, f'_1 is a p-th power, say $f'_1 \equiv h_0^p$. Thus, in T_k, $f'_1 = h_0^p + ap$, for some $a \in T_k$. Therefore, $f_1 = x_2 h_0^p + x_2 ap = h^p + bp^2$. It follows that, in T_k/pT_k, we have

$$0 \equiv h^p - x_2 h_0^p \equiv (h - \sqrt[p]{x_2}\, h_0)^p,$$

and thus, $h \equiv \sqrt[p]{x_2}\, h_0 \bmod pT_k$, so that $h = \sqrt[p]{x_2}\, h_0 + cp$ for some $c \in T_k$. Thus,

$$(\sqrt[p]{x_2}\, h_0 + cp)^p + bp^2 = x_2 h_0^p + a x_2 p.$$

Subtracting $x_2 h_0^p$ from both sides of this last equation, we have that p^2 divides the left-hand side of the equation, so p^2 divides the right-hand side of the equation, giving $p \mid a$. Thus, $f'_1 \in T_k^{p \wedge p^2}$, as required. We now note that the set $C := \{\sqrt[k]{x_2}, f'_1, f_2, \dots, f_r\}$ satisfies $\mathscr{A}_1$ in T_k. This follows since a height one prime in T_k contracts to a height one prime in S, and the set $\{x_2, f'_1, f_2, \dots, f_r\}$ satisfies $\mathscr{A}_1$ in S. Note, however, that for $j \geq 2$, the f_j need not be square-free in T_k, since $f_2, \dots, f_{t-1}$ are divisible by $x_2, \dots, x_t$, respectively.

Now, we repeat the process removing x_3 from the element f_2. The same argument as above shows $f_2' \in T_k^{p\wedge p^2} \cap S$ and $\{\sqrt[k]{x_2}, \sqrt[k]{x_3}, f_1', f_2', f_3, \dots, f_r\}$ satisfies $\mathscr{A}_1$ in T_k. Continuing this process we eventually arrive at the following set up:

(a) $\tilde{f}_1, \dots, \tilde{f}_r \in T_k^{p\wedge p^2} \cap S$ and no $\tilde{f}_j$ is divisible in S (or T_k) by any x_i.

(b) $\tilde{f}_j = f_j$ for $t \le j \le r$ and $f_j = x_j \tilde{f}_j$, where $\tilde{f}_j = f_j'$, for $1 \le j \le t-1$.

(c) The set $\{\sqrt[k]{x_2}, \dots, \sqrt[k]{x_t}, \tilde{f}_1, \dots, \tilde{f}_r\}$ satisfies $\mathscr{A}_1$.

We now observe that $\tilde{f}_1, \dots, \tilde{f}_r$ are square-free elements in T_k, and not divisible by p. The latter statement holds, since the $\tilde{f}_j$ are not divisible by p in S, and therefore are not divisible by p in T_k, since $pS = pT_k \cap S$. To see that $\tilde{f}_1, \dots, \tilde{f}_t$ are square-free in T_k, it suffices to show they are square-free in $T_k[1/p]$, since p does not divide any $\tilde{f}_j$. However, the set $\{\sqrt[k]{x_2}, \dots, \sqrt[k]{x_d}, \tilde{f}_1, \dots, \tilde{f}_r\}$ satisfies $\mathscr{A}_1$, so Proposition 2.2 gives us what we want.

For ease of notation, set $\sqrt[k]{x_i} := y_i$ for $2 \le i \le d$. We make one more ring extension. Set $\widetilde{T} := T_k[\sqrt[n_1]{y_1}, \dots, \sqrt[n_t]{y_t}]$ and let $\widetilde{E}$ denote its quotient field. Here we are choosing $\sqrt[n_i]{y_i}$ so that $(\sqrt[n_i]{y_i})^k = \sqrt[n_i]{x_i}$. Then $\widetilde{T}$ is an unramified regular local ring of mixed characteristic p, and since $\tilde{f}_1, \dots, \tilde{f}_r$ are square-free in T_k, satisfy $\mathscr{A}_1$, and each $\tilde{f}_j \in T_k^{p\wedge p^2}$, they remain such in $\widetilde{T}$. It follows that

$$F := \widetilde{E}\left(\sqrt[n_1]{\tilde{f}_1}, \dots, \sqrt[n_r]{\tilde{f}_r}\right)$$

is a class one radical extension of $\widetilde{E}$. Thus, by our assumption on the n_j, if we let $\widetilde{R}$ denote the integral closure of $\widetilde{T}$ in F, then Theorem 3.5 implies that $\widetilde{R}$ is Cohen–Macaulay. The proof of the theorem will be complete, once we observe that $R \subseteq \widetilde{R}$, and for this it suffices to see that $L(\omega_1, \dots, \omega_r) \subseteq F$.

To finish, we note that, since $f_j = x_{j+1}\tilde{f}_j$, for $1 \le j \le t-1$ and $f_j = \tilde{f}_j$, for $t \le j \le r$,

$$L(\omega_1, \dots, \omega_r) \subseteq L\left(\sqrt[n_1]{x_2}, \dots, \sqrt[n_t]{x_t}, \sqrt[n_1]{\tilde{f}_1}, \dots, \sqrt[n_r]{\tilde{f}_r}\right).$$

Since $(\sqrt[n_i]{y_i})^k = \sqrt[n_i]{x_i}$, $L\left(\sqrt[n_1]{x_2}, \dots, \sqrt[n_t]{x_t}, \sqrt[n_1]{\tilde{f}_1}, \dots, \sqrt[n_r]{\tilde{f}_r}\right) \subseteq F$, which gives what we want. $\square$

Remark 4.2. In Theorem 4.1, we can remove the restrictions that $f_1, \dots, f_r$ be square-free and satisfy $\mathscr{A}_1$, by allowing extra factors involving $x_2, \dots, x_d$. For example, if $f_1, \dots, f_r \in W(\underline{x})$, and each $f_j = m_j \tilde{f}_j$, where m_j is a monomial in $x_2 \dots, x_d$ and $\tilde{f}_1, \dots, \tilde{f}_r$ are square-free and satisfy $\mathscr{A}_1$, the same proof shows that $\tilde{f}_1, \dots, \tilde{f}_r$ belong to $W(\underline{x})$ and that there is a finite extension $\widetilde{T}$ of R such that $\widetilde{T}$ is an unramified regular local ring of mixed characteristic p with quotient field $\widetilde{E}$, so that

$$F := \widetilde{E}\left(\sqrt[n_1]{\tilde{f}_1}, \dots, \sqrt[n_r]{\tilde{f}_r}\right)$$

is a class one radical extension of $\widetilde{E}$.[4] Thus, the integral closure $\widetilde{R}$ of $\widetilde{T}$ in $\widetilde{E}$ is Cohen–Macaulay, and therefore a small Cohen–Macaulay algebra for R.

Corollary 4.3. *Let S be an unramified regular local ring of mixed characteristic p, and $g_1, \dots, g_s \in S$, not divisible by p (and not necessarily square-free) and suppose that in S we can write each $g_i := q_{i_1}^{c_{i_1}} \cdots q_{i_t}^{c_{i_t}}$ as a product of primes q_{i_j}. Let $\omega_i^{n_i} = g_i$ be roots in some field extension of L such that $p \mid n_i$ and $p^2 \nmid n_i$*

[4] In this scenario, $\tilde{f}_1, \dots, \tilde{f}_r$ will not be square-free and satisfy $\mathscr{A}_1$ until all x_i have been removed from all f_j.

for each i. Let R the integral closure of S in $K := L(\omega_1, \dots, \omega_r)$. Suppose there exists $W(\underline{x})$ as in Theorem 4.1 such that each $q_{i_j} \in W(\underline{x})$. Then R admits a small Cohen–Macaulay algebra.

Proof. We first point out that our elements g_j are products of primes and not a unit times a product of primes. We begin with the reduction used in the proof of Theorem 6.1 in [1], which we repeat for the reader's convenience. Write each $g_i := q_{i_1}^{d_{i_1} n_i + e_i} \cdots q_{i_t}^{d_{i_t} n_i + e_{i_t}}$. Then for

$$\gamma_i := q_{i_1}^{d_{i_1}} (\sqrt[n_i]{q_{i_1}})^{e_{i_1}} \cdots q_{i_t}^{d_{i_t}} (\sqrt[n_i]{q_{i_t}})^{e_{i_t}},$$

$\gamma_i^{n_i} = g_i$, so $S[\sqrt[n_i]{g_i}] \subseteq S[\sqrt[n_i]{q_{i_1}}, \dots, \sqrt[n_i]{q_{i_t}}]$, and hence $L(\sqrt[n_i]{g_i}) \subseteq L(\sqrt[n_i]{q_{i_1}}, \dots, \sqrt[n_i]{q_{i_t}})$. Doing this for each g_i shows K is contained in $F := L(\sqrt[n_1]{q_1}, \dots, \sqrt[n_r]{q_r})$. Thus, R is contained in the integral closure, say T, of S in F. Since $q_1, \dots, q_r$ are distinct primes in S, they are square-free and satisfy $\mathscr{A}_1$, and by assumption, they belong to V. Thus, by Theorem 4.1, T admits a small Cohen–Macaulay algebra, which in turn, is also a small Cohen–Macaulay algebra for R, which completes the proof. □

Corollary 4.4. *Let S be an unramified regular local ring of mixed characteristic p and $g_1, \dots, g_r \in S$, square-free and satisfying $\mathscr{A}_1$. Let $n_1, \dots, n_r$ be as in Theorem 4.1. Assume each $g_i = m_i + b_i p^2$, where m_i is a monomial in $x_2, \dots, x_d$ and $b_i \neq 0$. Then R admits a small Cohen–Macaulay algebra.*

Proof. If $h = x_2^{e_2} \cdots x_d^{e_d}$ is a monomial in $x_2, \dots, x_d$, then $m = \{(\sqrt[p]{x_1})^{e_1} \cdots (\sqrt[p]{x_d})^{e_d}\}^p$ in T_p. Thus, each $g_j \in W(\underline{x})$, and the result follows from Theorem 4.1. □

Example 4.5. The above results provide small Cohen–Macaulay algebras for many non-Cohen–Macaulay rings. Here is a concrete example. Consider Example 3.10 in [2]. We start with an unramified regular local ring of mixed characteristic 3, say S, and take $x, y \in S$ such that $3, x, y$ form part of a regular system of parameters for S. Take $a := xy^4 + 9$, $b := x^4 y + 9$, $f = ab^2$ and ω a cube root of f. If R is the integral closure of S in the quotient field of $S[\omega]$, then, as shown in [2], R is not Cohen–Macaulay. Set $T := S[\sqrt[3]{x}, \sqrt[3]{y}]$. Then T is an unramified regular local ring such that $3, \sqrt[3]{x}, \sqrt[3]{y}$ form a part of a minimal generating set of the maximal ideal of S. Then $a, b \in T$ are square-free, mutually coprime and lie in $T^{3\wedge 9}$. By Theorem 3.5, the integral closure of T in the quotient field of $T[\sqrt[3]{a}, \sqrt[3]{b}]$, say $\widetilde{R}$, is Cohen–Macaulay. Since $\widetilde{R}$ is a module finite extension of R, it is a small Cohen–Macaulay algebra for R.

We note two things about the constructions in this paper:

(i) In the above example, $f \in S^{3\wedge 9}$, but the integral closure of S in the quotient field of $S[\omega]$ is not Cohen–Macaulay. So, the square-free hypothesis in Theorem 3.4 cannot be dropped.

(ii) The integral closure in a general square-free p-th root tower with elements from $S^{p\wedge p}$ need not be Cohen–Macaulay. Consider Example 4.10 in [7]: Setting $S := \mathbb{Z}[X, Y]_{(p,X,Y)}$ for some prime number $p \geq 3$, take $f_1 := X^{2p} - pX^{2p} + p^2$ and $f_2 := (XY)^p + p(XY)^p + p^2$. Then $f_1, f_2 \in S^{p\wedge p} \setminus S^{p\wedge p^2}$ are square-free and mutually coprime. If R is the integral closure of S in $L(\sqrt[p]{f_1}, \sqrt[p]{f_2})$, then $\operatorname{proj\,dim}_S(R) = 1$.

Example 4.6. Here is an example illustrating Theorem 3.4. Take $S = \mathbb{Z}_3[\![X, Y]\!]$, where $\mathbb{Z}_3$ denotes the 3-adic integers and $f := X^3 + 9$, $g = Y^3 + 9$. Then $f, g \in S^{3\wedge 9}$ are square-free and mutually coprime. Let L denote the fraction field of S and let R be the integral closure of S in $L(\sqrt[3]{f}, \sqrt[3]{g})$. Then R is a free S-module of rank 9 with a basis given by $\mathscr{V} = \mathscr{V}_0 \cup \mathscr{V}_1 \cup \mathscr{V}_2$, where

$$\begin{aligned}
\mathscr{V}_0 &:= \{1, \sqrt[3]{f} - X, \sqrt[3]{g} - Y\},\\
\mathscr{V}_1 &:= \{3^{-1}(\sqrt[3]{f} - X)^2, 3^{-1}(\sqrt[3]{f} - X)(\sqrt[3]{g} - Y),\\
&\qquad 3^{-1}(\sqrt[3]{g} - Y)^2, 3^{-1}(\sqrt[3]{f} - X)(\sqrt[3]{g} - Y)^2, 3^{-1}([3]f - X)^2(\sqrt[3]{g} - Y)\},\\
\mathscr{V}_2 &= \{9^{-1}(\sqrt[3]{f} - X)^2(\sqrt[3]{g} - Y)^2\}.
\end{aligned}$$

5. Concluding remarks

In this section we make a few remarks concerning our assumptions, and how they might be successfully altered. As mentioned in the introduction, our primary assumption that the elements whose roots we take belong to $S^{p\wedge p^2}$ stems from the success we have had with this assumption in [3; 6; 7], together with the fact that there is an unramified prime over p with trivial extension of residue fields. In [2], for square-free f not a p-th power modulo pS, i.e., $f \notin S^{p\wedge p}$, it is noted that $S[\omega]$ is already integrally closed. Unfortunately, as shown in [6] and [7], adjoining p-th roots of even two square-free elements that are not in $S^{p\wedge p}$ presents considerable difficulties. In particular, if $\omega^p = f$ and $\mu^p = g$, then $S[\omega, \mu]$ need not be integrally closed, nor need its integral closure be Cohen–Macaulay, when S is regular. However, we can extend Theorem 3.4 by adjoining multiple elements that are not in $S^{p\wedge p}$ if we assume they are sufficiently independent modulo pS. Suppose that we start with a class one tower over S as in the statement of Theorem 3.4. and suppose that $S \hookrightarrow T$ is an integral extension of integrally closed domains such that T is S-free, $p \in T$ is a principal prime and the $f_i \in T$ remain square-free. Write E for the fraction field of T. Then applying Theorem 3.4 to T, the integral closure of T in $E(\sqrt[p]{f_1}, \dots, \sqrt[p]{f_r})$ is T-free. Therefore, the integral closure of S in $E(\sqrt[p]{f_1}, \dots, \sqrt[p]{f_r})$ is S-free. This is exactly what happens when we adjoin p-th roots of square-free elements *linearly disjoint modulo* p alongside a class one tower.

Definition 5.1. Let F denote the fraction field of S/pS. The elements $g_1, \dots, g_t$ in S are said to be *linearly disjoint modulo* p if $g_i \notin F^p$ for all i and there is an isomorphism of F-vector spaces

$$F\big(\sqrt[p]{\overline{g_1}}\big) \otimes_F \cdots \otimes_F F\big(\sqrt[p]{\overline{g_t}}\big) \simeq F\big(\sqrt[p]{\overline{g_1}}, \dots, \sqrt[p]{\overline{g_t}}\big),$$

where $\overline{g_i}$ is the image of g_i modulo p.

Note that if S/p is integrally closed, then $\bar{g} \notin F^p$ if and only if $g_i \notin S^p$. The following proposition extends the crucial result Theorem 3.4 from Section 3.

Proposition 5.2. *Let S be an integrally closed domain with fraction field L such that $p \in S$ is a nonzero principal prime. Let $f_1, \dots, f_r, g_1, \dots, g_t \in S$ be square-free elements satisfying $\mathscr{A}_1$ such that $L(\sqrt[p]{f_1}, \dots, \sqrt[p]{f_r})$ is a class one tower and the g_i are linearly disjoint modulo p. Then, the integral closure of S in $K := L(\sqrt[p]{f_1}, \dots, \sqrt[p]{f_r}, \sqrt[p]{g_1}, \dots, \sqrt[p]{g_t})$ is a free S-module. In particular, if S is Cohen–Macaulay or an unramified regular local ring, then the integral closure of S in K is Cohen–Macaulay.*

Proof. By Observations 2.3(ii), $T := S\big[\sqrt[p]{g_1}, \dots, \sqrt[p]{g_t}\big] \simeq S[X_1, \dots, X_t]/(X_1^p - g_1, \dots, X_t^p - g_t)$ where the X_i are indeterminates over S. The linear disjointness modulo p hypothesis on the g_i implies that $p \in T$ is a principal prime. By Observations 2.3(i), $T[1/p]$ is integrally closed. Since T is S-free, these facts imply that T is integrally closed. To conclude that the $f_i \in T$ are square-free, we can assume $p \in S$ is a unit, since $f_i \notin pS$ for all i. Applying Proposition 2.2 to $S[1/p]$, we see that the $f_i \in T$ are all square-free. Thus the f_i define a class one tower over T and hence by Theorem 3.4, the integral closure of T in $L(\sqrt[p]{f_1}, \dots, \sqrt[p]{f_r}, \sqrt[p]{g_1}, \dots, \sqrt[p]{g_t})$ is a free T-module and hence a free S-module. □

Remark 5.3. With Proposition 5.2 in hand, it is not difficult to see that Theorem 3.5 can be extended in a similar way.

When S is a complete unramified regular local ring of mixed characteristic $p > 0$ whose residue field is F-finite, there is an alternate way of dealing with elements that do not belong to $S^{p\wedge p}$, as the following proposition shows.

Proposition 5.4. *Let $(S, \mathfrak{m}, k)$ be a complete unramified regular local ring of mixed characteristic $p > 0$ with k F-finite. Then there exists an unramified regular local ring T of mixed characteristic $p > 0$ such that T is finite over S and $S \hookrightarrow T^{p\wedge p}$.*

Proof. Complete p to a minimal system of generators of $\mathfrak{m}$, say $\mathfrak{m} = (p, x_2, \dots, x_d) = (p, \underline{x})$, and let x_i' denote the image of x_i in S/pS. Let F denote the Frobenius map on S/pS. By hypothesis, $E := S/pS$ is an F-finite regular local ring and $E^{1/p}$ is obtained by adjoining to E the p-th roots of the x_i' and the p-th roots of a basis of k over k^p. Take T to be the S-algebra obtained by adjoining the p-th roots of the x_i and the p-th roots of a fixed set of lifts of a basis of k over k^p. By construction, $S \hookrightarrow T^{p\wedge p}$. Moreover, it follows easily that T is regular local with maximal ideal generated by $(p, \sqrt[p]{x_2}, \dots, \sqrt[p]{x_d})$ and residue field $k^{1/p}$.[5] □

Remark 5.5. Let S, $\underline{x}$ and T be as in the preceding proposition, so that $S \subseteq T^{p\wedge p}$. Assume further that the residue field k is perfect. Thus, $T = T_p(\underline{x})$, where for $k \geq 1$, $T_k(\underline{x})$ has the same meaning as in Theorem 4.1. We want to observe, that in this case, $T_p(\underline{x}) \cap S = \bigcup_{k\geq 1}(T(\underline{x})^{p\wedge p^2} \cap S) = W(\underline{x})$. Suppose $f = h^p + ap^2$, with $h, a \in T_k(\underline{x})$, for some $k > p$, and $f \in S$ square-free. We can write $f = h_0^p + bp$, for $h_0, b \in T$. Then $0 \equiv h^p - h_0^p \equiv (h - h_0)^2 \bmod pT_k(\underline{x})$, from which it follows that $h \equiv h_0 \bmod pT_k(\underline{x})$. Thus, $h = h_0 + cp$, for some $c \in T_k(\underline{x})$. Therefore $f = (h_0 + cp)^p + ap^2 = h_0^p + dp^2$, for some $d \in T_k(\underline{x})$. Thus $f - h_0^p \in p^2T_k(\underline{x}) \cap T = p^2T$, since T is integrally closed and $T_k(\underline{x})$ is integral over T. It follows that $f \in T^{p\wedge p^2}$, which is what we want.

Consider the question of the existence of maximal Cohen–Macaulay modules over the integral closure of a complete regular local ring S of mixed characteristic $p > 0$ with perfect residue field, in an arbitrary p-th root tower over its quotient field K. One of the main differences between our point of view in this paper and our earlier ones is that we do not restrict ourselves to *birational* Cohen–Macaulay modules or algebras (see Theorem 4.1), whereas the constructions in [2; 6; 7] are all birational. Thus, results like Proposition 5.4 simultaneously allow us to move beyond the birational assumption and also enable us to restrict attention to the case of general square-free towers with elements chosen from S^p. Indeed, choose an unramified regular local ring T such that $S \hookrightarrow T^{p\wedge p}$ as in Proposition 5.4. Let L' be the fraction field of T and set $\mathscr{K} := K[L']$. Note that Proposition 2.2 implies that square-free elements of S that are not divisible by the regular system of parameters used to construct T from S remain square-free in T. It is then clear that $\mathscr{K}$ embeds in a finite field extension of it, say $\widetilde{\mathscr{K}}$, obtained by adjoining p-th roots of mutually coprime square-free elements of T coming from $T^{p\wedge p}$ to L'. Let $\mathscr{R}$ be the integral closure of S in $\widetilde{\mathscr{K}}$. It then suffices to show that $\mathscr{R}$ admits a maximal Cohen–Macaulay module. In summary,

[5] That T is regular can be checked one element at a time. After adjoining the p-th roots of the x_i, we get a URLR. When we adjoin the p-th roots of the units in question, the generators of the maximal ideal remain the same at each step.

in attempting to construct a maximal Cohen–Macaulay module or algebra over R, we can simply start over and assume that the elements whose roots we adjoin are mutually coprime, square-free and come from $S^{p\wedge p}$.

Finally we address the question of units. As pointed out in the introduction, when the exponents n_i are units, the ring $S[\sqrt[n_1]{f_1}, \dots, \sqrt[n_r]{f_r}]$ is integrally closed and a free S-module. It is not difficult to show that if, say, in Theorem 3.5 we allow some of the n_i to be units, while the other $n_i = pd_i$ with d_i a unit, then the conclusion of the theorem still holds. The conclusions of our other results then follow in this case as well. However, we have focused on the case that the n_i are nonunits, since we are motivated by the case where S is an unramified regular local ring of mixed characteristic p, and the extensions of quotient fields have degree divisible by p. We could also consider the case that some of the f_i are units. However, this potentially introduces roots of unity and we lose the property that the degree of the extension of quotients field equals p^r. We address the issue involving roots of unity in a forthcoming paper [4].

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