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Complexes acyclic up to integral closure

BY D. KATZ
Department of Mathematics, University of Kansas, Lawrence, KS 66045, *U.S.A.*

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Introduction

In [**R**] D. Rees introduced the notions of reduction and integral closure for modules over a commutative Noetherian ring and proved the following remarkable result. Let R be a locally quasi-unmixed Noetherian ring and I an ideal generated by n elements. Suppose that height $(I) = h$. Then the ith module of cycles in the Koszul complex on a set of n generators for I is contained in the integral closure of the ith module of boundaries for $i > n-h$. This result should be considered a dimension-theoretic analogue of the famous depth sensitivity property of the Koszul complex demonstrated by Serre and Auslander–Buschsbaum in the 1950s. At roughly the same time, Hoschster and Huneke introduced the notion of tight closure and thereafter gave a number of theorems in the same (though considerably broader) vein for tight closure. In particular, in [**HH**] they showed that if R is an equidimensional local ring of characteristic $p > 0$, which is a homomorphic image of a Gorenstein ring, then for all $i > 0$, the ith module of cycles is contained in the tight closure of the ith module of boundaries for any complex satisfying the so-called standard rank and height conditions (see the definitions below). Since the tight closure is contained in the integral closure for such rings, the result of Hochster and Huneke extends (in characteristic p) considerably the result of Rees. In fact, their result could be considered a dimension–theoretic analogue of the Buchsbaum–Eisenbud exactness theorem ([**BE**]), which in a certain sense is the ultimate depth sensitivity theorem. Moreover, using the technique of reduction to characteristic p, Hochster and Huneke have shown that their results hold in equicharacteristic zero as well, whenever the tight closure is defined.

Unfortunately, as is often the case, the situation for non-equicharacteristic rings is not very well understood. Indeed, if one could find the right sort of analogue to the Buchsbaum–Eisenbud exactness theorem, a number of the unsettled homological conjectures in commutative algebra would fall. It is the purpose of the present paper to provide further examples of complexes (apart from Koszul complexes) satisfying the property that for all $i > 0$, the ith module of cycles is contained in the integral closure of the ith module of boundaries (without making any characteristic assumption on the ring). We shall say that such a complex is *acyclic up to integral closure*. Our main result states that if R is a quasi-unmixed local ring whose completion is Cohen–Macaulay on its punctured spectrum, then any complex of free modules satisfying the standard rank and height conditions is acyclic up to integral closure provided the entries of the matrices associated to the maps in the complex lie in a sufficiently large power of the maximal ideal. As a corollary we deduce that if R is any three dimensional quasi-unmixed local ring, then there exists an $N > 0$, such that any complex which satisfies the standard rank and height conditions and whose matrices have entries in the Nth power of the maximal ideal is acyclic up to integral closure. Examples of such complexes are easy to come by. One may take a 'generically acyclic' complex in a polynomial ring in three variables over the integers and replace the variables in the complex by an appropriately chosen system of parameters. The details are given below.

We now briefly describe the contents of the present paper. In Section 1, we recall relevant definitions and establish notation. In particular, we recall the definition of the integral closure of a module given by Rees, as well as describing in more detail the complexes to be considered. In Section 2, we prove our main result, along with a few ancillary results. For example, we give an alternate proof of the theorem of Rees mentioned above for ideals of the principal class. Finally, in Section 3, we present a couple of sufficient conditions for the sort of complexes we consider to be acyclic up to integral closure over an arbitrary (quasi-unmixed) local ring.

Section 1

Throughout, R will be a local ring with maximal ideal $\mathfrak{m}$. Many of the hypotheses and conclusions of our results can be phrased for non-local rings, simply because the properties we consider are preserved under localization. We have opted for less generality in favour of expositional ease. Our principal objects of study will be bounded complexes of finitely generated free R modules, i.e. complexes of the form

$$\mathbf{F}: \quad 0 \longrightarrow F_n \xrightarrow{\phi_n} \dots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow 0,$$

where F_i is a free R module of rank b_i and ϕ_i is a $b_{i-1} \times b_i$ matrix with entries in R. The rank of ϕ_i is the size of the largest non-vanishing minor of ϕ_i and we shall write $I(\phi_i)$ for the ideal generated by minors of size $\operatorname{rank}(\phi_i)$. Furthermore, unless stated otherwise, we shall assume that each ϕ_i has entries in $\mathfrak{m}$. For each $i > 0$ we write $Z_i(\mathbf{F})$, $B_i(\mathbf{F})$ and $H_i(\mathbf{F})$ respectively for $\ker(\phi_i)$, $\operatorname{im}(\phi_{i+1})$ and $Z_i(\mathbf{F})/B_i(\mathbf{F})$. In accordance with [**HH**], the complex $\mathbf{F}$ is said to satisfy the *standard rank and height conditions* (henceforth abbreviated *srhc*) if:

(*i*) $\operatorname{rank}(\phi_i) + \operatorname{rank}(\phi_{i+1}) = \operatorname{rank}(F_i)$, and
(ii) $\operatorname{height}(I(\phi_i)) \geqslant i$, for $i = 1, \dots, n$.

Though we shall have little need for the terminology, we recall that the complex $\mathbf{F}$ is said to satisfy the *standard rank and depth conditions* (abbreviated *srdc*) if conditions (i) and (ii) hold with $\operatorname{depth}(I(\phi_i))$ replacing $\operatorname{height}(I(\phi_i))$ in (ii). With this terminology, the main theorem of [**BE**] asserts that $\mathbf{F}$ is acyclic if and only if $\mathbf{F}$ satisfies *srdc*. Thus, the point of the results referred to in [**HH**], [**R**] and those presented here may be viewed as attempts at describing the cycles in complexes satisfying *srhc* as opposed to *srdc*. There are numerous ways that complexes satisfying *srhc* can arise, so we pause briefly to describe a few. First of all, suppose $n = \dim(R)$ in $\mathbf{F}$ and $H_i(\mathbf{F})$ has finite length for all $i \geqslant 0$. Then it readily follows that $\mathbf{F}$ satisfies *srhc*. (Note that by the New Intersection Theorem (see [**PR1**]), if $\mathbf{F}$ has finite length homology, $n \geqslant \dim(R)$. On the other hand, if $\mathbf{F}$ satisfies *srhc*, then $n \leqslant \dim(R)$, so the equality $n = \dim(R)$ is forced.) If $S \subseteq R$ is an extension of Noetherian rings for which $\text{height}(J) = \text{height}(JR)$ for all ideals $J \subseteq S$, then $\mathbf{F} = \mathbf{G} \otimes_S R$ satisfies *srhc* whenever $\mathbf{G}$ is acyclic over S. More concretely, one may specialize a generically acyclic complex at a system of parameters (see [**EH**]). To elaborate, let $\mathbf{x} = x_1, \ldots, x_d$ be a system of parameters and make R into a $B = \mathbf{Z}[X_1, \ldots, X_d]$ module by sending X_i to x_i. Assume that S is a regular local subring over which R is finite, whose maximal ideal is generated by $(\mathbf{x})$. Let $\mathbf{G}$ be an acyclic complex over $\mathbf{Z}[X_1, \ldots, X_d]$ with $\mathbf{Z}$-flat, graded augmentation. By the main result of [**EH**], $\mathbf{G} \otimes_B S$ is acyclic, so by the remarks above $\mathbf{G} \otimes_B R$ satisfies *srhc*. For example, if J is any ideal generated by monomials in $\mathbf{x}$ and I is the ideal of $\mathbf{Z}[X_1, \ldots, X_d]$ generated by the corresponding monomials in the X_i, then tensoring any resolution for I with R yields a complex satisfying *srhc*. We shall return briefly to this type of construction below.

We would now like to recall the definition of the integral closure of an R module. We begin by recalling the definition of the integral closure of an ideal. An element $x \in R$ is said to be integrally dependent on the ideal I, if x satisfies an equation of the form

$$x^r + i_1 x^{r-1} + \ldots + i_r = 0, \quad \text{with} \quad i_j \in I^j, \quad \text{for} \quad i = i, \ldots, r.$$

The set of elements integrally dependent on I form an ideal called the integral closure of I, which we shall denote by I_a. If R is a domain then $I_a = \cap\, IV$, where the intersection ranges over all discrete valuation domains between R and its quotient field. This is the characterization that Rees builds upon in [**R**] to extend the notion of integral closure to modules.

Definition 1·1. Let R be a Noetherian domain with quotient field K and $M \subseteq R^h$, a finitely generated R module. The integral closure of M (in R^h) is the R module $(\cap MV) \cap R^h$, where as above, V ranges over the discrete valuation domains between R and K.

Here we are identifying $R^h \subseteq V^h \subseteq K^h$ and think of MV as the V submodule of V^h generated by M. We shall denote the integral closure of M in R^h by M_a and suppress the reference to R^h when convenient. One can extend the definition of M_a to the non-domain case by defining M_a to be the elements of R^h which map to $(M(R/\mathrm{p})^h)_a$ for each minimal prime ideal p.

Definition 1·2. Let $\mathbf{F}$ be a bounded complex of finitely generated free R modules. $\mathbf{F}$ is said to be acyclic up to integral closure if for all $i > 0$, $Z_i(\mathbf{F}) \subseteq (B_i(\mathbf{F}))_a$.

In the following proposition we record several equivalent conditions for a complex satisfying the standard rank condition (i.e. condition (i) in the definition of *srhc*) to be acyclic up to integral closure. In particular, the proposition implies that we may check for acyclicity up to integral closure by passing to the completion of R and modding out minimal primes. Before stating the proposition, we need a bit more notation. An element $v \in R^h$ may be regarded as a column vector and as such corresponds in obvious fashion to a linear form in the polynomial ring $R[U_1, \ldots, U_h]$. We write $l(v)$ for this linear form. For $M \subseteq R^h$, we write $L(M)$ for the ideal generated by the forms $l(v)$, $v \in M$.

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Proposition 1·3. *Suppose that* **F** *satisfies the standard rank condition and height* $(I(\phi_i)) > 0$, *for all* $i > 0$. *The following conditions are equivalent*:

(i) **F** *is acyclic up to integral closure*;
(ii) $\mathbf{F} \otimes (R/\mathrm{p})$ *is acyclic up to integral closure for all minimal prime ideals* p;
(iii) $\mathbf{F} \otimes (R/(\mathit{nilrad}\,(R))$ *is acyclic up to integral closure*;
(iv) *Statements* (i), (ii) *and* (iii) *hold with* R^* *replacing* R, *for* R^* *the completion of* R;
(v) $L(Z_i(\mathbf{F})) \subseteq (L(B_i(\mathbf{F})))_a$, *for* $i > 0$;
(vi) *If* R *is reduced*, $Z_i(\mathbf{F}) = (B_i(\mathbf{F}))_a$, *for* $i > 0$.

Proof. The equivalence of statements (i), (ii) and (iii) follows from Definitions 1·1 and 1·2. Suppose for the moment R is a domain with quotient field K. Since **F** satisfies the standard rank condition, the complex becomes exact when tensored with K. Thus $Z_i(\mathbf{F})K = B_i(\mathbf{F})K$, for all $i > 0$. By [**R**, theorem 1·5], it follows that $Z_i(\mathbf{F})$ is integral over $B_i(\mathbf{F})$ if and only if $L(B_i(\mathbf{F}))R[L(Z_i(\mathbf{F}))]$ is an irrelevant ideal of $R[L(Z_i(\mathbf{F}))]$, where $R[L(Z_i(\mathbf{F}))]$ denotes the subring of $R[U_1, \ldots, U_h]$ generated over R by the linear forms generating $L(Z_i(\mathbf{F}))$. (Here, $h = \mathrm{rank}\,(F_i)$.) It is not difficult to see that this latter condition holds if and only if in $R[U_1, \ldots, U_h]$, the ideal $L(Z_i(\mathbf{F}))$ is integral over the ideal $L(B_i(\mathbf{F}))$. Thus (i) and (v) are equivalent when R is a domain. Since each $I(\phi_i)$ has height greater than zero, the standard rank condition holds modulo each minimal prime. Therefore, the equivalence of (i) and (v) in general follows from the domain case, the definitions, and the fact that elements of a ring are integral over an ideal if and only if their images modulo each minimal prime are integral over the image of the ideal. Using statement (v) and faithful flatness, one can see that the statements in (iv) are equivalent to statement (i). To finish the proof, it suffices to see that $(B_i(\mathbf{F}))_a \subseteq Z_i(\mathbf{F})$ always holds when R is reduced, and for this it suffices to assume that R is a domain. Suppose $v \in (B_i(\mathbf{F}))_a$. Then $l(v)$ is integral over $L(B_i(\mathbf{F}))$ and hence integral over $L(Z_i(\mathbf{F}))$. But $L(Z_i(\mathbf{F}))$ is generated by the linear forms in the kernel of the ring homomorphism from $R[U_1, \ldots, U_h]$ to $R[L(B_{i-1}(\mathbf{F}))]$ determined by ϕ_i. Since this kernel is prime, $l(v)$ belongs to it and hence to $L(Z_i(\mathbf{F}))$. Hence $v \in Z_i(\mathbf{F})$.

As mentioned above, if R is a d dimensional local ring, finite over a regular local subring S (e.g. R is a complete local domain) and $x_1, \ldots, x_d$ is a system of parameters which generates the maximal ideal of S, then certain generically defined complexes 'evaluated' at the x_i satisfy *srhc*. We show in the next proposition that this holds for every system of parameters in a quasi-unmixed local ring. Recall that R is said to be quasi-unmixed if its completion is equi-dimensional.

Proposition 1·4. *Suppose that* R *is a* d *dimensional quasi-unmixed local ring and* $x_1, \ldots, x_d$ *form a system of parameters. Let* $B = \mathbf{Z}[X_1, \ldots, X_d]$ *and make* R *into a* B *module by sending* X_i *to* x_i. *Let* **F** *be an acyclic complex of finitely generated free* B-*modules with graded*, **Z**-*flat augmentation. Then* $\mathbf{F} \otimes_B R$ *satisfies srhc.*

Proof. Since R is quasi-unmixed, it is well-known that for any ideal $J \subseteq R$, height (J) equals the minimum of the heights of the ideals $(JR^* + \mathrm{p}/\mathrm{p})$, for R^* the completion of R and p a minimal prime of R^*. It follows that $\mathbf{F} \otimes_B R$ satisfies *srhc* if and only if $\mathbf{F} \otimes_B (R^*/\mathrm{p})$ satisfies *srhc* for every minimal prime ideal $\mathrm{p} \subseteq R^*$, so we may assume that R is a complete local domain. Suppose that we could find a d-dimensional Cohen–Macaulay local ring S and a minimal prime $Q \subseteq S$ such that $R = S/Q$ and the pre-images of the x_i form a system of parameters in S. Then by the main result of [**EH**], $\mathbf{F} \otimes_B S$ is an acyclic complex, so $\mathbf{F} \otimes_B S$ satisfies *srdc*. Since S

is quasi-unmixed, it follows from the foregoing that $\mathbf{F} \otimes_B R$ satisfies *srhc*. To see the existence of S, we use the Cohen structure theorem. While it is well-known that we can find a d-dimensional Cohen–Macaulay (in fact, complete intersection) local ring mapping onto R, we want to observe that this can be done so that the pre-image of the x's remains a system of parameters.

To this end, let T be an unramified regular local ring and $P \subseteq T$ a prime ideal satisfying $R = T/P$. We write I for the ideal generated by a set of pre-images of the x_i in T. Then $\dim(T) = \text{height}(I+P) \leqslant \text{height}(I) + \text{height}(P)$, by Serre's theorem, so $d = \dim(R) = \dim(T) - \text{height}(P) \leqslant \text{height}(I)$. By Krull's principal ideal theorem, $\text{height}(I) \leqslant d$, so $\dim(T) = d + \text{height}(P)$. Since $I+P$ is primary for the maximal ideal of T, it follows that (working mod I) we may find elements $y_1, \ldots, y_s \in P$ such that $(x_1, \ldots, x_d, y_1, \ldots, y_s)T$ is primary for the maximal ideal of T, where $s = \text{height}(P)$. Thus, taken together the x's and the y's form a maximal regular sequence in T. We now take $S = T/(y_1, \ldots, y_s)T$ and $Q = P/(y_1, \ldots, y_s)T$.

We want to close this section by mentioning a conjecture which underscores the importance of investigating properties of complexes satisfying *srhc*. This conjecture is an integral closure analogue of the Buchsbaum–Eisenbud exactness theorem. Though it has not been formally named (as far as I know), it has been discussed in private conversation for a number of years. (Personally, I have discussed the conjecture or aspects of it with C. Huneke and P. Roberts). The conjecture is a theorem for local rings containing a field by the work of Hochester and Huneke (and, as mentioned above, one has the same conclusion but with tight closure replacing integral closure, whenever the former is defined).

Conjecture. Let R be a quasi-unmixed local ring and $\mathbf{F}$ a complex satisfying *srhc*. Then $\mathbf{F}$ is acyclic up to integral closure.

An immediate consequence of this conjecture would be a positive solution to the monomial conjecture of M. Hochster (and hence a positive solution to a number of well-known conjectures in commutative algebra). We indicate briefly the implication. Let R be a d dimensional local ring and $x_1, \ldots, x_d$ a system of parameters. The monomial conjecture asserts that $(x_1 \ldots x_d)^k \notin (x_1^{k+1}, \ldots, x_d^{k+1})R$ for all $k > 0$. By passing to the completion and modding out a prime ideal of maximal dimension, one may assume that R is a complete local domain, hence quasi-unmixed. If the monomial conjecture fails then one has an equation of the form

$$a_1 x_1^{k+1} + \ldots + a_d x_d^{k+1} - x_1^k \ldots x_d^k = 0$$

holding in R. Let $B = \mathbf{Z}[X_1, \ldots, X_d]$ and $\mathbf{G}$ be the acyclic complex providing a minimal resolution for the ideal $(X_1^{k+1}, \ldots, X_d^{k+1}, X_1^k \ldots X_d^k)B$. If the conjecture above holds, then $\mathbf{F} = \mathbf{G} \otimes_B R$ is acyclic up to integral closure. It follows that the one-cycle $(a_1, \ldots, a_d, -1)^{\mathrm{t}}$ ('t' for transpose) is integral over $B_1(\mathbf{F})$. However, one can show that the last row of the matrix corresponding to $B_1(\mathbf{F})$ has entries in $(x_1, \ldots, x_d)R$. It would then follow that $1 \in (x_1, \ldots, x_d)_a$, which cannot be. As we will have occasion to refer to this complex throughout this paper, for future reference we call this complex (or any complex derived from it by specializing a system of parameters) the MC complex. If $d = 3$ and x, y, z form a system of parameters, then the MC complex has the form

$$\mathbf{F}: \quad 0 \longrightarrow R^3 \xrightarrow{\phi_3} R^6 \xrightarrow{\phi_2} R^4 \xrightarrow{\phi_1} R \longrightarrow 0,$$

where $\phi_1 = (x^{k+1}\ y^{k+1}\ z^{k+1}\ x^k\ y^k\ z^k)$ and

$$\phi_2 = \begin{pmatrix} -y^{k+1} & -z^{k+1} & 0 & -y^kz^k & 0 & 0 \\ x^{k+1} & 0 & -z^{k+1} & 0 & -x^kz^k & 0 \\ 0 & x^{k+1} & y^{k+1} & 0 & 0 & -x^ky^k \\ 0 & 0 & 0 & x & y & z \end{pmatrix},$$

$$\phi_3 = \begin{pmatrix} 0 & 0 & -z^k \\ -y^k & 0 & 0 \\ 0 & -x^k & 0 \\ z & 0 & y \\ 0 & z & -x \\ -x & -y & 0 \end{pmatrix}.$$

Thus, while the monomial conjecture asserts no MC complex has a one-cycle of the form $(a, b, c, -1)^t$, the conjecture above would assert that every one-cycle is integral over the submodule of R^4 generated by the columns of ϕ_2.

Section 2

In this section we prove the results stated in the introduction which allow us to give characteristic free examples of complexes that are acyclic up to integral closure. We begin by giving an alternate proof to the theorem of Rees concerning Koszul complexes, in the special case where the elements in question are part of a system of parameters. Before doing so, we record a fundamental observation, due to Ratliff, concerning relations on subsets of systems of parameters in quasi-unmixed local rings (see [**LR1**, theorem 2·12]).

Ratliff's fundamental observation. Let R be a quasi-unmixed local ring and $x_1, \dots, x_n$ a subset of a system of parameters (i.e. height $(I) = n$, for $I = (x_1, \dots, x_n)R$). For all $i \geqslant 0$, if $rx_{i+1} \in (x_1, \dots, x_i)R$, then $r \in ((x_1, \dots, x_i)R)_a$.

PROPOSITION 2·1 (cf. [**R**; Theorem 3·1]). *Let R be a quasi-unmixed local ring and $x_1, \dots, x_n$ a subset of a system of parameters. Write*

$$\mathbf{K}: \quad 0 \longrightarrow K_n \xrightarrow{\psi_n} \dots \xrightarrow{\psi_2} K_1 \xrightarrow{\psi_1} K_0 \longrightarrow 0$$

for the Koszul complex on $x_1, \dots, x_n$. Then **K** *is acyclic up to integral closure.*

Proof. We use Ratliff's fundamental observation in conjunction with well-known properties of the Koszul complex. Since the image of $x_1, \dots, x_n$ remains part of a system of parameters modulo any minimal prime (a well-known property of quasi-unmixed local rings), by Proposition 1·3 we may assume that R is a domain. Fix $i > 0$ and set $I = (x_1, \dots, x_n)R$. Suppose $\psi_i(z) = 0$. Then we have an R linear combination of the columns of ψ_i equal to zero whose coefficients are the coordinates of z. Each row of this vector equation gives rise to an equation of the form

$$z_{i_1} x_{i_1} + \dots + z_{i_r} x_{i_r} = 0.$$

Hence each z_{i_j} is integral over an ideal generated by a subset of the x's and therefore integral over I. Moreover, the structure of the matrix ψ_i makes it clear that each coordinate of z appears in such an equation, hence the coordinates of z belong to I_a.

One of the other hand, it is well known that for each j, multiplication by x_j kills the homology of **K**. In fact,

$$x_j z = \pm z_{j_1} C_{j_1} \pm \ldots \pm z_{j_s} C_{j_s},$$

where $C_{j_1}, \ldots, C_{j_s}$ are the columns of ψ_{i+1} involving x_j and $z_{j_1}, \ldots, z_{j_s}$ are the corresponding entries of z. Thus $IZ_i(\mathbf{K}) \subseteq I_a B_i(\mathbf{K})$. Hence $I(Z_i(\mathbf{K})\,V) \subseteq I_a(B_i(\mathbf{K})\,V)$ for each discrete valuation domain V contained in the quotient field of R. Since $IV = I_a V$ is principal, we obtain $Z_i(\mathbf{K})\,V \subseteq B_i(\mathbf{K})\,V$. That is, the ith module of cycles is integral over the ith module of boundaries, as desired.

Remark 2·2. Let R be a quasi-unmixed local ring and $I \subseteq R$ an ideal minimally generated by the elements $x_1, \ldots, x_n$. Suppose that height$(I) = h$. We would like to indicate how the previous argument can be used to prove Rees's theorem in its full generality. In other words, letting **K** as above denote the Koszul complex on $x_1, \ldots, x_n$, we want to see that $Z_i(\mathbf{K}) \subseteq B_i(\mathbf{K})_a$ for $i > n-h$. As in the proof of Proposition 2·1, it suffices to show that $IZ_i(\mathbf{K}) \subseteq I_a B_i(\mathbf{K})$ for $i > n-h$. Moreover, since Koszul complexes on different minimal generating sets for I are isomorphic, it follows that we are free to change minimal generating sets (as the condition $IZ_i(\mathbf{K}) \subseteq I_a B_i(\mathbf{K})$ holding for one Koszul complex will hold for any other Koszul complex isomorphic to it). Therefore, by standard general position arguments (i.e. prime avoidance arguments) we may further assume that any subset $x_{i_1}, \ldots, x_{i_h}$ consisting of h elements of the x_i's generates an ideal having height h. Suppose now that $i > n-h$ and $\psi_i(z) = 0$. As before, we consider this to be a dependence relation on the columns of ψ_i with the coordinates of z as coefficients. Again, each row of this vector equation gives rise to an equation involving coordinates of z and $n-i+1$ of the x's. Since $i > n-h$, $n-i+1 \leqslant h$. Thus, Ratliff's fundamental observation applies and we may argue as before to see that $Iz \subseteq I_a B_i(\mathbf{K})$.

Our next result discusses the situation for complexes **F** satisfying *srhc* for local rings of dimension less than or equal to three. The results for dimension less than or equal to two are rather easy, but we include them for the sake of completeness. The result for dimension three can be obtained from the theorem below, but contains in rather pure form the essence of the more general argument, so isolating it seems worthwhile.

PROPOSITION 2·3. *Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring and* **F** *as above a complex satisfying srhc. Then*

(i) *if* $\dim(R) = 1$ *or* $\dim(R) = 2$, *then* **F** *is acyclic up to integral closure*;
(ii) *if* $\dim(R) = 3$, *there exists* $N > 0$, *such that if each matrix ϕ_i has its entries in $\mathfrak{m}^N$, then* **F** *is acyclic up to integral closure.*

Proof. Since R is quasi-unmixed, for any ideal $J \subseteq R$, height(J) equals the minimum of the heights of the ideals $(JR^* + \mathfrak{p}/\mathfrak{p})$, for R^* the completion of R and $\mathfrak{p}$ a minimal prime of R^*. Hence $\mathbf{F} \otimes (R^*/\mathfrak{p})$ satisfies *srhc* for each $\mathfrak{p}$, so we may assume that R is a complete local domain (using Proposition 1·3). If $\dim(R) = 1$, $n \leqslant 1$, so the complex is (now) acyclic. If $\dim(R) = 2$, then **F** is exact at F_2. Let $(R', \mathfrak{m}')$ denote the integral closure of R. Then R' is Cohen–Macaulay, so $Z_1(\mathbf{F})\,R' = B_1(\mathbf{F})\,R'$. Hence, for each discrete valuation domain V, $Z_1(\mathbf{F})\,V = B_1(\mathbf{F})\,V$, so **F** is acyclic up to integral closure. Suppose now that $\dim(R) = 3$. If $n \leqslant 2$, the preceding arguments show that **F** is acyclic up to integral closure. If $n = 3$, again by the same arguments, we have

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that $Z_i(\mathbf{F})$ is integral over $B_i(\mathbf{F})$ for $i = 2, 3$. In fact, we may pass to R' and assume that $H_i(\mathbf{F}) = 0$, for $i = 2, 3$. Thus, R' is a complete 3-dimensional local domain such that R'_P is Cohen–Macaulay for all non-maximal prime ideals. In this situation it is well-known that the annihilator of the second local cohomology module $H^2_{m'}(R')$ is m'-primary and that the first homology of the complex $\mathbf{F}$ has finite length. Let x_1, x_2, x_3 be a system of parameters contained in this annihilator and assume that N is large enough so that $m^N \subseteq (x_1, x_2, x_3)R' = I$. We now assume each ϕ_i has entries in I. By Roberts' theorem, [**PR2**] (or its generalizations [**S**], [**HH**], or [**W**]), multiplication by I kills $H_1(\mathbf{F})$. Let $z \in Z_1(\mathbf{F})$. By the comparison theorem, we have a commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & F_3 & \xrightarrow{\phi_3} & F_2 & \xrightarrow{\phi_2} & F_1 & \longrightarrow & F_1/B_1 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \theta_3} & & \uparrow{\scriptstyle \theta_2} & & \uparrow{\scriptstyle \theta_1} & & \uparrow{\scriptstyle \theta_0} & & \uparrow & & \\
0 & \longrightarrow & K_3 & \xrightarrow{\psi_3} & K_2 & \xrightarrow{\psi_2} & K_1 & \xrightarrow{\psi_1} & K_0 & \longrightarrow & R'/(x_1,\, x_2,\, x_3)\, R' & \longrightarrow & 0,
\end{array}$$

where the bottom row is the Koszul complex on x_1, x_2, x_3 and the maps θ_i are induced by θ_0 which takes 1 to z. Thus there are column vectors v_1, v_2, v_3 in F_2 such that $\phi_2(v_i) = x_i z$ and the v_i form the columns of θ_1. Let a, b, c be the entries of any row of θ_2. Since $\theta_2 \psi_3 = 0$, $ax_3 - bx_2 + cx_1 = 0$. By Ratliff's fundamental observation, a, b, $c \in I_a$. Hence the entries of $\phi_3 \theta_2$ belong to II_a. Therefore the entries of $\theta_1 \psi_2$ belong to II_a. Suppose the first row of θ_1 has entries a_1, b_1, c_1. Then, since the $(1, 1)$ entry of $\theta_1 \psi_2$ belongs to II_a, we have an equation

$$-a_1 x_2 + b_1 x_1 = h_1 x_1 + h_2 x_2 + h_3 x_3$$

with $h_i \in I_a$. Bringing the left hand side to the right, we obtain a relation on the x_i from which we deduce that a_1, $b_1 \in I_a$. Similarly, we see that the remaining entries of θ_1 belong to I_a. Thus each v_i has its coordinates in I_a and we obtain a vector relation $Iz \subseteq I_a \operatorname{im}(\phi_2)$. As in the proof of Proposition 2·1, this shows that z is integral over $B_1(\mathbf{F})$ as desired.

Remark 2·4. Let R be a 3-dimensional quasi-unmixed local ring and x, y, z a system of parameters. As mentioned in the previous section, the monomial conjecture asserts that for all $k > 0$, the complex MC associated to the ideal $(x^{k+1}, y^{k+1}, z^{k+1}, x^k y^k z^k)R$ has no one-cycles of the form $(a, b, c, -1)^t$. In [**H**], Hochster has shown that the conclusion of the monomial conjecture holds for the system of parameters x^m, y^m, x^m for $m \gg 0$. Hence the complex associated to $(x^{mk+m}, y^{mk+m}, z^{mk+m}, (xyz)^{mk})R$, has no one-cycle of the form $(a, b, c-1)^t$. The proposition above strengthens this conclusion, in that it shows that every cycle in the latter complex is integral over the corresponding module of boundaries. Of course Hochster's result holds in arbitrary dimension. It would be extremely interesting to have the conclusion of statement (iii) in Proposition 2·3 hold in arbitrary dimension as well. At this point, we are confined to local rings Cohen–Macaulay on their punctured spectrum, since we need to be able to choose a system of parameters in an appropriate 'universal annihilator'. We proceed to do this in the following theorem.

THEOREM 2·5. *Let (R, m) be a quasi-unmixed local ring whose completion is Cohen–Macaulay on its punctured spectrum. Then there exists an $N > 0$ with the following property. If $\mathbf{F}$ is any complex satisfying srhc and each matrix ϕ_i has its entries in m^N, then $\mathbf{F}$ is acyclic up to integral closure.*

Proof. Let **F** as above be a complex satisfying *srhc*. Using Proposition 1·3, we may assume that R is complete and therefore Cohen–Macaulay on its punctured spectrum. In particular, the homology of **F** has finite length in degree greater than zero. Let A denote the product of the annihilators of the local cohomology modules $H^i_{\mathfrak{m}}(R)$ for $i = 0, \ldots, \dim(R)-1$. The hypotheses on R insure that A is $\mathfrak{m}$ primary and again, by Roberts' theorem ([**PR2**], [**S**], [**HH**], or [**W**]), multiplication by A kills $H_i(\mathbf{F})$, for $i > 0$. We would like to compare the complex **F** with the Koszul complex on a system of parameters from A in a manner analogous to what was done in the proof of Proposition 2·3. However, for a given $i > 0$, the truncated complex ending in $F_i/B_i(\mathbf{F})$ need not be exact. Nevertheless, we can extend an initial map as before, if we raise certain elements in the system of parameters to large enough powers. This type construction is spelled out in greater generality as Lemma 9·16 in [**HH**].

To begin, let $I = (x_1, \ldots, x_d)R$ be an ideal generated by a system of parameters contained in A. Unlike the proof of Proposition 2·3, we cannot require only that the entries of the ϕ_i belong to I. Some of these maps must have entries in I^2, but we shall see precisely which ones. In fact, suppose $H_i(\mathbf{F}) = 0$, for $i = s, \ldots, n$ and that the entries of ϕ_i belong to I^2 for $1 \leqslant i \leqslant s$ and to I for $s+1 \leqslant i \leqslant n$. By induction on n (the length of the complex) we may assume that $Z_i(\mathbf{F}) \subseteq (B_i(\mathbf{F}))_a$, for $i > 1$. (We may repeat the arguments given in the proof of the first part of Proposition 2·3 to handle the base cases $n = 1, 2$.) Now, let $z \in Z_1(\mathbf{F})$ and select any n of the x_i, say $x_1, \ldots, x_n$. As above, there exist column vectors $v_1, \ldots, v_n$ such that $\phi_2(v_i) = x_i z$. Thus we may find θ_0, θ_1 which start a map of complexes

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & F_3 & \xrightarrow{\phi_3} & F_2 & \xrightarrow{\phi_2} & F_1 & \longrightarrow & F_1/B_1 & \longrightarrow & 0 \\
 & & & \uparrow{\scriptstyle\theta_1} & & \uparrow{\scriptstyle\theta_0} & & \uparrow & & \\
\cdots \longrightarrow & K_2 & \xrightarrow{\psi_2} & K_1 & \xrightarrow{\psi_1} & K_0 & \longrightarrow & R/(x_1, \ldots, x_n)R & \longrightarrow & 0,
\end{array}$$

where the bottom row is the Koszul complex on $x_1, \ldots, x_n$, $\theta_0(1) = z$, and θ_1 takes the standard basis of K_1 to the x_i. In order to find a map θ_2, $\theta_1\psi_2$ must map to $\operatorname{im}(\phi_3)$. Of course, it need not. It does however map to $\ker(\phi_2)$. Since multiplication by x_1 multiplies $\ker(\phi_2)$ into $\operatorname{im}(\phi_3)$ we can multiply θ_0 and θ_1 by x_1 and find a map θ_2 such that the diagram

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & F_3 & \xrightarrow{\phi_3} & F_2 & \xrightarrow{\phi_2} & F_1 & \longrightarrow & F_1/B_1 & \longrightarrow & 0 \\
 & \uparrow{\scriptstyle\theta_2} & & \uparrow{\scriptstyle\theta_1 x_1} & & \uparrow{\scriptstyle\theta_0 x_1} & & \uparrow & & \\
\cdots \longrightarrow & K_2 & \xrightarrow{\psi_2} & K_1 & \xrightarrow{\psi_1} & K_0 & \longrightarrow & R/(x_1, \ldots, x_n)R & \longrightarrow & 0
\end{array}$$

commutes. If we continue in this manner, we can find maps θ_j for $j = 0, \ldots, s-1$ so that the diagram

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & F_s & \xrightarrow{\phi_s} & F_{s-1} & \longrightarrow \cdots \longrightarrow & F_2 & \xrightarrow{\phi_2} & F_1 & \xrightarrow{\phi_1} & F_1/B_1 & \longrightarrow & 0 \\
 & \uparrow{\scriptstyle\theta_{s-1}} & & \uparrow{\scriptstyle\theta_{s-2}x_1} & & \uparrow{\scriptstyle\theta_1 x_1^{s-2}} & & \uparrow{\scriptstyle\theta_0 x_1^{s-2}} & & \uparrow & & \\
\cdots \longrightarrow & K_{s-1} & \xrightarrow{\psi_{s-1}} & K_{s-2} & \longrightarrow \cdots \longrightarrow & K_1 & \xrightarrow{\psi_1} & K_0 & \longrightarrow & R/(x_1, \ldots, x_n)R & \longrightarrow & 0
\end{array}$$

commutes. Since **F** is exact at F_j for $j \geqslant s$, there is no problem in finding the appropriate θ_j's which extend from the rest of the Koszul complex to **F**. Using

Ratliff's fundamental observation, as in the proof of Proposition 2·3, we see that each θ_j for $s-1 \leqslant j \leqslant n$ has entries in I_a. Since ϕ_s has entries in I^2 and $\phi_s \theta_{s-1} = (\theta_{s-2} x_1) \psi_{s-1}$, it follows that the entries of $(\theta_{s-2} x_1) \psi_{s-1}$ belong to $I^2 I_a$. Fixing one such entry, it follows that we may write a quadratic equation in the generators of I with coefficients from I_a on the left hand side and coefficients from θ_{s-2} on the other. Observe that the monomials of degree two in the x_i appearing in the right hand side of this equation are distinct. Bringing one side of this equation to the other, we obtain a quadratic relation on the generators of I. By [**LR2**, (4·14·2)] the coefficients in such a relation must lie in I_a. Hence any entry of θ_{s-2} appearing in such an equation must be in I_a. Since all of the entries of θ_{s-2} will appear in such an equation, they all belong to I_a. At the next square in the diagram, $\phi_{s-1}(\theta_{s-2} x_1) = (\theta_{s-3} x_1^2) \psi_{s-2}$. Since ϕ_{s-1} has entries in I^2 and θ_{s-2} has entries in I_a, we can, in a similar manner, obtain a cubic relation on the generators of I, from which we deduce that θ_{s-3} has entries in I_a (again, using [**LR2**, (4·142)]). Continuing in this way, we eventually arrive at the conclusion that θ_1 has coefficients in I_a. Thus $(x_1, \ldots, x_n) z \subseteq I_a B_1(\mathbf{F})$. If we now repeat the argument as often as necessary, using the remaining generators of I (but always taking subsets of length n), we conclude $Iz \subseteq I_a B_1(\mathbf{F})$. Since this relation holds modulo each minimal prime, we may 'cancel' I from both sides over any discrete valuation domain (as before), to conclude $z \in (B_1(\mathbf{F}))_a$, as desired.

Remark 2·6. (i) The proof of the theorem shows that for all complexes $\mathbf{F}$ whose maps have entries in a sufficiently large power of the maximal ideal, $IZ_i(\mathbf{F}) \subseteq I_a B_i(\mathbf{F})$, so this is a stronger conclusion than the 'uniform annihilation' granted by Roberts' theorem (or theorems of a similar ilk in [**HH**]). Of course (at present) the condition is applicable to a considerably smaller class of complexes. However, it seems doubtful that a similar argument along the lines above that is more 'complex specific' is readily available. In other words, given a complex $\mathbf{F}$, for any natural choice of I killing the homology, it is usually not the case that the entries of the maps in the complex belong to I. For example, if one takes the MC complex determined by the ideal $(x^{k+1}, y^{k+1}, z^{k+1}, x^k y^k z^k) R$, it is well-known that $x^{k+1}, y^{k+1}, z^{k+1}$ kill the homology of the complex, but the entries of the complex do not lie in the ideal they generate. On the other hand, the entries of the complex clearly lie in the ideal generated by x, y, z, but these need not kill the homology (in general).

(ii) In [**W**], C. Wickham has shown that if R is a local ring admitting a dualizing complex, then some power of A (the product of the local cohomological annihilators) kills the homology in every complex satisfying *srhc*. (Actually a much subtler statement regarding the powers of the annihilators is given.) Thus, by choosing I generated by a subset of a system of parameters contained in an appropriate power of A, we may repeat the argument given in the proof of Theorem 2·5, to see that complexes $\mathbf{F}$ (satisfying *srhc*) whose maps have entries either in I or I^2 are acyclic up to integral closure.

(iii) We can get a bit more mileage from the arguments presented in Proposition 2·3 and Theorem 2·5 by using the following lovely result due to Itoh (see [**I**]). Let R be a quasi-unmixed local ring and I an ideal generated by a subset of a system of parameters. Then $(I^{n+1})_a \cap I^n = I_a I^n$ for all $n \geqslant 1$. Using this result, we may repeat the arguments above requiring only that the entries of the maps appearing in the complex belong to I_a or $(I^2)_a$. In particular, we obtain the following corollary.

COROLLARY 2·7. *Let (R, m) be a complete quasi-unmixed local ring of dimension d. Suppose that depth$(R) = d-1$ and R satisfies Serre's condition S_{d-1}. Let $A = ann(H_m^{d-1}(R))$ and suppose $A_a = m$. Then, any complex* **F** *satisfying srhc is acyclic up to integral closure.*

Proof. Without loss of generality we may assume that the residue field of R is infinite. Let I be a minimal reduction of A, and therefore m. Then arguing as before, using the remark above, it follows that $IZ_1(\mathbf{F}) \subseteq I_a B_1(\mathbf{F})$ for any complex **F**, since all maps have entries in m and therefore I_a. Since the conditions of the corollary imply the only non-zero homology can occur at F_1, this proves the corollary.

Remark 2·8. If (R, m) is a complete three dimensional local ring with depth$(R) = 2$ satisfying Serre's condition S_2 and $ann(H_m^2(R)) = m$ (i.e. R is Buchsbaum), then Roberts' theorem applied to any MC complex shows that the monomial conjecture holds for R. The Corollary above improves this somewhat in that it shows that the monomial conjecture holds if one assumes only that the annihilator reduces the maximal ideal.

The last result of this section shows again the advantage of having maps whose entries belong to certain annihilating ideals. In the proposition below we extend an argument due to L. Burch from the setting of ideals with finite projective dimension to the setting of complexes and annihilators of homology. (See [**B**], theorem 5.)

PROPOSITION 2·9. *Let* **F** *(as above) be a complex of finitely generated free R modules such that the homology of* **F** *is non-zero only at F_1 and F_0. Let $I = ann(H_1(\mathbf{F}))$. Suppose there exists an R module L such that the* nth *and* $n+1$st *maps in the minimal resolution for L are non-zero and have their entries in I and I_a, respectively. Then $IZ_1(\mathbf{F}) \subseteq I_a B_1(\mathbf{F})$. In particular, the cycles in* **F** *are integrally dependent on the boundaries.*

Proof. Let

$$\mathbf{G}: \quad \ldots \longrightarrow G_2 \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} G_0 \longrightarrow L \longrightarrow 0$$

be a minimal resolution of L and set $M = \text{coker}(\phi_2)$. Since M has projective dimension $n-1$, $\text{Tor}_n(M, L) = \text{Tor}_{n+1}(M, L) = 0$. Let $z \in Z_1(\mathbf{F})$ and e be the first standard basis element of G_n. Since ψ_n has entries in I, $(\psi_n \otimes 1_M)(e \otimes z') = 0$, where z' denotes the image of z in M. Thus $e \otimes z' = (\psi_{n+1} \otimes 1_M)(w')$, for $w' \in G_{n+1} \otimes M$. Thus $e \otimes z = (\psi_{n+1} \otimes 1_{F_1})(w) + b_1$ in $G_n \otimes F_1$, for some $b_1 \in \text{im}(1_{G_n} \otimes \phi_2)$. Let $i \in I$. Then $i(e \otimes z) = (\psi_{n+1} \otimes 1_{F_1})(iw) + ib_1$, so $(\psi_{n+1} \otimes 1_{F_1})(iw) \in \text{im}(1_{G_n} \otimes \phi_2)$. Therefore $(\psi_{n+1} \otimes 1_M)(iw') = 0$. Consequently $iw = (\psi_{n+2} \otimes 1_{F_1})(u) + b_2$, for $u \in G_{n+2} \otimes F_1$ and $b_2 \in \text{im}(1_{G_{n+1}} \otimes \phi_2)$. Thus

$$e \otimes iz = (\psi_{n+1} \otimes 1_{F_1})(iw) + ib_1 = (\psi_{n+1} \otimes 1_{F_1})((\psi_{n+2} \otimes 1_{F_1})(u) + b_2) + ib_1,$$

hence $e \otimes iz = (\psi_{n+1} \otimes 1_{F_1})(b_2) + ib_1$. Since ψ_{n+1} has entries in I_a, if we interpret this last equation in terms of **F**, we obtain $Iz \subseteq I_a B_1(\mathbf{F})$, as desired.

Section 3

In this section we offer a couple of observations regarding the possibility that every complex satisfying *srhc* over a quasi-unmixed local ring is acyclic up to integral closure. In the first statement below, we consider the situation for generically defined complexes evaluated at a system of parameters. Should this statement hold, Proposition 3·2 implies that any (minimal) resolution of an ideal generically defined by monomials becomes acyclic up to integral closure when evaluated at a system of parameters. Of course this would have as an immediate consequence the validity of M. Hochster's monomial conjecture. It follows from Proposition 3·4 below that should our second statement hold, then the conjecture stated at the end of Section 1 is valid.

Let $\mathbf{F}$ as above be a complex satisfying *srhc*. In the proof of Proposition 2·3 and Theorem 2·5 we found an ideal I with the property that $IZ_i(\mathbf{F}) \subseteq I_a B_i(\mathbf{F})$, and saw that this implied that the complex $\mathbf{F}$ was acyclic up to integral closure. Since there exists a $c > 0$ such that $I(I_a)^c = (I_a)^{c+1}$, it follows that if we let $J = (I_a)^{c+1}$, then $JZ_i(\mathbf{F}) = JB_i(\mathbf{F})$ for all $i > 0$. Whenever there exists an ideal J with this property, we shall say that J efficiently kills the homology of $\mathbf{F}$. It follows readily that every complex $\mathbf{F}$ satisfying *srhc* will be acyclic up to integral closure, if we can find an ideal which efficiently kills its homology. We now consider the following statement.

Statement 3·1. *Let R be a Noetherian ring and*

$$0 \longrightarrow \mathbf{K} \longrightarrow \mathbf{G} \longrightarrow \mathbf{L} \longrightarrow 0$$

an exact sequence of bounded complexes of finitely generated free R modules. Assume that there exists an ideal J_1 efficiently killing the homology of $\mathbf{K}$ *and an ideal J_2 efficiently killing the homology of* $\mathbf{L}$. *Then there exists an ideal I efficiently killing the homology of* $\mathbf{G}$.

Suppose now that R is a d dimensional quasi-unmixed local ring and $x_1, \ldots, x_d$ is a system of parameters. Let $B = \mathbf{Z}[X_1, \ldots, X_d]$ and make R into a B module by sending X_i to x_i. Let N be a graded $\mathbf{Z}$-flat B module admitting a filtration $0 = N_0 \subseteq N_1 \subseteq \ldots \subseteq N_s = N$ such that each $N_{i+1}/N_i = \mathbf{Z}$ (for example, N could be B/H for any ideal H generated by monomials in the X_i such that H contains some power of each X_i). Finally, let $\mathbf{G}$ be any resolution of N over B. Then we have:

Proposition 3·2. *If Statement* 3·1 *holds, then $\mathbf{G} \otimes_B R$ is acyclic up to integral closure.*

Proof. As before, we are free to assume that R is a complete local domain. Induct on s the number of terms in the filtration of N. When $s = 1, N = \mathbf{Z}$. If $\mathbf{G}$ is the Koszul complex on the X_i, then $\mathbf{G} \otimes_B R$ is the Koszul complex on the x_i, and the result follows from Proposition 2·1. If $\mathbf{G}$ is not the Koszul complex on the X_i, then we can employ the argument that follows to compare $\mathbf{G} \otimes_B R$ with the Koszul complex on the x_i in a manner that allows us to conclude that $\mathbf{G} \otimes_B R$ is acyclic up to integral closure. Now assume $s > 1$. There exists an exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow N \longrightarrow M \longrightarrow 0$$

with M a $\mathbf{Z}$-flat graded B module admitting a length $s-1$ filtration with factors isomorphic to $\mathbf{Z}$. It follows that there exists an exact sequence of resolutions (over B)

$$0 \longrightarrow \mathbf{K} \longrightarrow \mathbf{G}' \longrightarrow \mathbf{L} \longrightarrow 0,$$

where $\mathbf{K}$ is the Koszul complex on the X_i, $\mathbf{L}$ is a resolution of M and $\mathbf{G}'$ is a resolution of N obtained in standard fashion from $\mathbf{K}$ and $\mathbf{L}$ (so each $G'_j = K_j \oplus L_j$). By the proof of Proposition 2·1, the comments above and induction on s, there exist ideals J_1 and J_2 which (respectively) efficiently kill the homology of $\mathbf{K} \otimes_B R$ and $\mathbf{L} \otimes_B R$. Since

$$0 \longrightarrow \mathbf{K} \otimes_B R \longrightarrow \mathbf{G}' \otimes_B R \longrightarrow \mathbf{L} \otimes_B R \longrightarrow 0$$

is an exact sequence of complexes, by Statement 3·1 there exists an ideal I which efficiently kills the homology of $\mathbf{G}' \otimes_B R$. We need to relate $\mathbf{G}' \otimes_B R$ to $\mathbf{G} \otimes_B R$. To this end, let S (as in the proof of Proposition 1·4) be a d-dimensional complete intersection local ring mapping onto R in which the pre-images of the x's form a system of parameters. By the main result of [**EH**], $\mathbf{G} \otimes_B S$ and $\mathbf{G}' \otimes_B S$ are (not necessarily minimal) resolutions of $N \otimes_B S$. Using standard facts about minimal resolutions, we can find split exact complexes $\mathbf{F}$ and $\mathbf{F}'$ of finitely generated free S modules such that $(\mathbf{G} \otimes_B S) \oplus \mathbf{F} = (\mathbf{G}' \otimes_B S) \oplus \mathbf{F}'$. Thus $(\mathbf{G} \otimes_B R) \oplus (\mathbf{F} \otimes_S R) = (\mathbf{G}' \otimes_B R) \oplus (\mathbf{F}' \otimes_S R)$. Since $\mathbf{F} \otimes_S R$ and $\mathbf{F}' \otimes_S R$ remain split exact, it follows readily that I efficiently kills the homology of $(\mathbf{G}' \otimes_B R) \oplus (\mathbf{F}' \otimes_S R)$ and from this that I efficiently kills the homology of $\mathbf{G}$.

The second statement we wish to consider is a dimension-theoretic analogue of the Auslander–Buchsbaum formula. The formula states that if M is a finitely generated module with finite projective dimension over a Noetherian local ring R, then

$$\operatorname{depth}(M) + \text{proj. dim.}(M) = \operatorname{depth}(R).$$

It follows that if $\mathbf{F}$ (as above) is a complex of finitely generated free R modules which is exact at F_j for $j \geqslant 2$, then for $P \in \operatorname{Ass}(F_1/B_1(\mathbf{F}))$, $\operatorname{depth}(R_P) \leqslant n-1$. We offer a possible analogue for complexes satisfying *srhc*, and record immediately the ramifications of such a statement.

Statement 3·3. *Let R be a quasi-unmixed local ring and $\mathbf{F}$ a complex satisfying srhc. Suppose $Z_i(\mathbf{F}) = B_i(\mathbf{F})_a$ for $i \geqslant 2$ and $P \in \operatorname{Ass}(F_1/B_1(\mathbf{F})_a)$. Then height $(P) \leqslant n-1$.*

PROPOSITION 3·4. *Let R be a quasi-unmixed local ring. If Statement* 3·3 *holds, then every complex satisfying srhc is acyclic up to integral closure.*

Proof. Without loss of generality, we may assume that R is a domain. We proceed by induction on n, the length of the complex. If $n = 1$ or $n = 2$, we may repeat the argument given in the proof of the first part of Proposition 2·3 to see that $\mathbf{F}$ is acyclic up to integral closure. Suppose $n \geqslant 3$. By induction, we need only show $Z_1(\mathbf{F})$ is integral over $B_1(\mathbf{F})$. Since $\mathbf{F}$ tensored with the quotient field of R is exact, there exists $0 \neq \lambda \in R$ with $\lambda Z_1(\mathbf{F}) \subseteq B_1(\mathbf{F}) \subseteq B_1(\mathbf{F})_a$. Thus, either $Z_1(\mathbf{F}) \subseteq B_1(\mathbf{F})_a$ or $\lambda \in P$ for some $P \in \operatorname{Ass}(F_1/B_1(\mathbf{F})_a)$. Suppose the latter condition holds. By Statement 3·3, height $(P) \leqslant n-1$. Localize at P. Since height $(I(\phi_n)) \geqslant n$, $I(\phi_n)_P = R_P$. Hence, im $(\phi_n)_P$ is a summand of $(F_{n-1})_P$ and we may split off this irrelevant term to obtain a complex $\mathbf{F}'$ of length $n-1$. By induction, $\mathbf{F}'$ is acyclic up to integral closure. Hence, $F'_1/(B_1(\mathbf{F}'))_a = F'_1/Z_1(\mathbf{F}')$ is a torsion-free R_P module. Thus $P = 0$, so $\lambda = 0$, contradiction.

Remark 3·5. Regarding Statement 3·1, it is a simple matter to check that if J_1 and J_2 respectively kill the homology of $\mathbf{K}$ and $\mathbf{L}$, then $J_1 J_2$ kills the homology of $\mathbf{G}$. Needless to say, it does not seem to follow that $J_1 J_2$ efficiently kills the homology of $\mathbf{G}$ if J_1 and J_2 efficiently kill the homology of $\mathbf{K}$ and $\mathbf{L}$. Perhaps a refinement of the spectral sequence arguments given in [**PR2**] or [**W**] will do the trick, but efforts along these lines have eluded the present author. Regarding Statement 3·3, in order to settle the monomial conjecture it would be sufficient to verify the statement for $\mathbf{F} = \mathbf{G} \otimes_S R$, for R a complete integrally closed local domain, S a regular local subring and $\mathbf{G}$ the MC complex on a regular system of parameters. In fact, as Proposition 3·4 shows, for $P \in \operatorname{Ass}(F_1/B_1(\mathbf{F})_a)$, height $(P) = 0$, once it is known that height $(P) \leqslant \dim(R) - 1$. The former would follow at once if one could show that $P \cap S \in \operatorname{Ass}(G_1/B_1(\mathbf{G})_a)$. For then, $\mathbf{G}$ exact implies $B_1(\mathbf{G})_a = B_1(\mathbf{G})$, so height $(P \cap S) = 0$.

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