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UNIFORM SYMBOLIC TOPOLOGIES IN ABELIAN EXTENSIONS

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Abstract. In this paper we prove that, under mild conditions, an equicharacteristic integrally closed domain which is a finite abelian extension of a regular domain has the uniform symbolic topology property.

1. Introduction.

The purpose of this paper is to give a partial answer to the following question:

Question 1.1. *Let R be a complete local Noetherian domain. Does there exist a positive integer b such that for all prime ideals $P \subseteq R$, $P^{(bn)} \subseteq P^n$, for all $n \geq 1$?*

In fact, our results do not in general need the assumption that the ring is local or complete, but do require the rings to satisfy both the Uniform Artin-Rees and the Uniform Briançon-Skoda properties.

Here we write $P^{(t)}$ to denote the t^{th} symbolic power of the prime ideal P, namely $P^{(t)} = P^tR_P \cap R$. For any Noetherian domain R, when b as above exists, we shall say that R satisfies the *uniform symbolic topology property* on prime ideals. Uniform results of this type for regular rings were first given by Ein, Lazarsfeld and Smith in [5], by Hochster and Huneke in [8], and recently by Ma and Schwede in mixed characteristic [17]. In these papers, the authors prove that if R is a regular local ring and d is the Krull dimension of R, then $P^{((d-1)n)} \subseteq P^n$, for all prime ideals $P \subseteq R$ and all $n \geq 1$. In [15], uniform results were proved for isolated singularities, under some mild conditions on the ring. However, in that paper no effective bound was given. In general little is known: see the introduction to [15] for further discussion about this problem. Because a complete local domain containing a field, or an affine domain over a field, is a finite extension of a finite dimensional regular domain containing a field, it is natural to consider how the uniform symbolic topology property behaves with respect to finite ring extensions. Thus, Question 1.1 would have a positive answer for such rings if whenever $S \subseteq R$ is a finite extension of Noetherian domains, R has the uniform symbolic topology property on prime ideals if S has the uniform symbolic topology property on prime ideals. In [16] ascent and descent theorems of this type were proved. Although descent of the uniform symbolic

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topology property holds, the results in [16] for ascent are not strong enough to give a positive answer to Question 1.1.

The entire paper is devoted to a proof of the following main theorem: Suppose that R is an integrally closed domain that is an abelian extension of an equicharacteristic excellent regular domain S satisfying our standard hypothesis (see the next section), such that if S has characteristic $p > 0$, then S is F-finite and the index of the corresponding Galois group is not divisible by p. Then R has the uniform symbolic topology property on prime ideals.

Preliminary results and basic definitions are contained in Section 2.

There are many delicate points in the proof. Section 3 sets up a main technical tool for the proof, which holds in great generality. Namely, we prove that for a wide class of rings, the uniform symbolic topology property holds for all prime ideals in R if there exist *fixed* integers $a, b \geq 1$, with b a particular value chosen *a priori*, such that $P^{(a)} \subseteq P^b$, for all prime ideals P. The number b depends on a uniform Artin-Rees number for certain special elements which we call uniform multipliers for symbolic powers. This already presents a difficulty in using reduction to characteristic p, since it is not known how uniform Artin-Rees numbers behave under such reduction.

Section 4 gives our main results in the case of a simple radical extension of an excellent regular ring satisfying our standard hypothesis. The main new technical tool is a fundamental result involving norms of elements in a simple radical extension. For a given element u in a finite extension ring of our base ring, and for a given prime Q, our result compares which symbolic power of the prime Q the element u is in to which symbolic power its norm is in for the contraction of Q to the base ring. This section also presents the reduction to characteristic p argument to prove certain elements are always uniform multipliers for symbolic powers.

Section 5 generalizes the simple radical extension case to the case of repeated radical extensions. For this, after some preliminary results, we rely on induction and the existence of a uniform multipler for symbolic powers in repeated radical extensions. Our final Section 6 combines the previous work to prove the main theorem. We use Kummer theory in the following way. Suppose that $S \subseteq R$ is a finite abelian extension of integrally closed Noetherian domains. In other words, if L denotes the quotient field of S and K denotes the quotient field of R, then the extension $L \subseteq K$ is a Galois extension with abelian Galois group and R is the integral closure of S in K. If the characteristic of L (say) does not divide order of the Galois group and L contains an appropriate root of unity, then K is an extension of the form $L(\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_t})$. It is not hard to see that if S is regular (or just a UFD), we may assume each a_i is a square-free element S, that the ring $T := S[\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_t}]$ is integrally closed, and $R \subseteq T$. Thus, by the descent theorem stated in the next section, it will be enough to show that T satisfies the uniform symbolic topology property.

For a more detailed history of the problem at hand, we refer the reader to [15] or [16] and for unexplained terminology, we refer the reader to the book [4]. The paper [19] contains interesting related results concerning the linear equivalence of topologies defined by valuation ideals.

2. Preliminaries

In this brief section we record the results that we will rely upon throughout the paper. Our work relies heavily on both the Uniform Artin-Rees Property and the Uniform Briançon-Skoda Property. Because of this dependence, many of our theorems need to assume we are in a position to use them. This leads to the following definition:

Definition 2.1. Throughout this paper, we say that a reduced Noetherian ring S satisfies our *standard hypothesis* if for every finite extension T of S and reduced ideal $J \subseteq T$, T/J satisfies both the Uniform Artin-Rees Property and the Uniform Briançon-Skoda Property.

For the reader's convenience, we recall the definitions:

Definition 2.2. Let S be a Noetherian ring. We say that the *Uniform Artin-Rees Property* holds for S if for every pair of finitely generated R-modules $N \subseteq M$, there exists an integer k (depending on $N \subseteq M$) such that for all ideals I of S, and for all $n \geq k$,

$$I^n M \cap N \subseteq I^{n-k} N.$$

Definition 2.3. Let S be a Noetherian reduced ring. We say that the *Uniform Briançon-Skoda Property* holds if there exists a positive integer k such that for all ideals I of S, and for all $n \geq k$,

$$\overline{I^n} \subseteq I^{n-k}.$$

Here we are writing $\overline{J}$ to denote the integral closure of an ideal J.

Let S be a reduced Noetherian ring. By [11, Theorems 4.12, 4.13], in each of the following cases, S satisfies our standard hypothesis.

i) S is essentially of finite type over an excellent Noetherian local ring.
ii) S is a ring of characteristic p, and under the Frobenius map $F : S \to S$, S is a finite module over the image of the Frobenius. If S is reduced, this is equivalent to saying that $S^{1/p}$ is module finite over S.
iii) S is essentially of finite type over $\mathbf{Z}$.

Two main results of [16], which we use freely in this paper, are the ascent and descent theorems mentioned in the introduction. Note that in [16], the ring S is *acceptable* if it satisfies one of the three conditions above. In fact, the results below hold when S satisfies our standard hypothesis, since in [16] we used the acceptable hypothesis in order to invoke the Uniform Artin-Rees and the Uniform Briançon-Skoda properties.

Theorem 2.4. *(Ascent) Let $S \subset R$ be a finite integral extension of Noetherian domains. Assume that S is acceptable and integrally closed in its field of fractions K and that the field of fractions L of R is a separable extension of K. If S has the uniform symbolic topology property on prime ideals, then R has the uniform symbolic topology property for all prime ideals $Q \subseteq R$ such that Q is the only prime lying over $Q \cap S$. Moreover, if R is also integrally closed, the conclusion holds for arbitrary L.*

Theorem 2.5. *(Descent) Let $S \subset R$ be a finite integral extension of Noetherian domains. Assume that S is acceptable and integrally closed. There exists an integer r, depending only on the extension $S \subset R$, such that if Q is a prime in R, $q = S \cap Q$, and $Q^{(bn)} \subset Q^n$, for some fixed b and for all $n \geq 1$, then $q^{(rbn)} \subset q^n$ for all $n \geq 1$. In particular, if R satisfies the uniform symbolic symbolic topology property, then so does S.*

As a consequence of the descent theorem, for example, if a finite group acts on an acceptable regular ring of equicharacteristic zero having finite Krull dimension, then the ring of invariants must have uniform symbolic topologies for prime ideals.

3. Uniform multipliers for symbolic powers and Bootstrapping

In this section we show that for a wide class of rings the uniform symbolic topology property holds for all prime ideals in R, if there exist *fixed* integers $a, b \geq 1$, with b a particular value chosen *a priori*, such that $P^{(a)} \subseteq P^b$, for all prime ideals P. Of course, on the face of it, this property (which is implied by the uniform symbolic topology property), is much weaker. A key ingredient in this result is the existence of certain elements that multiply large symbolic powers of an ideal into powers of smaller symbolic powers. To this end, we make the following definition.

Definition 3.1. Let R be a Noetherian ring and U a set of ideals of R (for example, all prime ideals or all reduced ideals). We say that a non-zerodivisor $x \in R$ is a *uniform multiplier for symbolic powers with respect to U* if there exists $k \geq 1$ such that for all ideals $I \in U$,

$$x^n I^{(kn+en)} \subseteq (I^{(e+1)})^n,$$

for all $e \geq 0$ and $n \geq 1$. If $U = \operatorname{Spec}(R)$, we just say that x is a uniform multiplier for symbolic powers. In either case, we refer to the integer k as the *index* of the multiplier x.

Remark 3.2. It follows from [8], Theorem 1.1, that if R is a finite-dimensional regular domain containing a field, then $1 \in R$ is a uniform multiplier for symbolic powers for all ideals. The same theorem shows that if R is a geometrically reduced affine domain over a field K (which in the case that R has characteristic zero, just means that R is reduced), then any x in the square of the Jacobian ideal of R over K is a uniform multiplier for symbolic powers for all ideals. In [15], Proposition 3.4, it is shown that if R is is a Noetherian domain containing a field of characteristic $p > 0$ such that R is F-finite and an isolated singularity, then exists an $\mathfrak{m}$-primary ideal consisting uniform multipliers for symbolic powers. In the lemma below, we point out how the argument from [15], which is quite similar to the one in [8], yields uniform multipliers for symbolic powers in more general settings. In particular, the existence of a uniform multiplier for symbolic powers in a repeated radical extension of an equicharacteristic regular ring is one of our crucial results. In [14], we prove the existence of these uniform multipliers for arbitrary hypersurfaces.

Lemma 3.3. *Suppose R is a d-dimensional F-finite integral domain containing a field of characteristic $p > 0$. Fix $a \in R$ and assume there are flat R-modules $F_q \subseteq R^{1/q}$ such that $aR^{1/q} \subseteq F_q$, for all q. Then for every ideal $I \subseteq R$, if we let h denote the maximum of the analytic spreads of the ideals I_P, where P is an associated prime of I, then $a^n I^{(nh+en)}$ is contained in the tight closure of $(I^{(e+1)})^n$,* *for all $e \geq 0$ and $n \geq 1$. Moreover, if a is also a test element for R, then a^2 is a uniform multiplier for symbolic powers for all ideals of R with index d.*

Proof. With only very minor modifications, we can follow the proofs of Proposition 3.4 and Theorem 3.5 in [15] to show that a has the required property. Starting with an ideal $I \subseteq R$, as in [15] Proposition 3.4, one uses the flatness of the F_q to show that $a^q(I^{[q]} : x^q) \subseteq (I : x)^{[q]}$, for all $I \subseteq R$, $x \in R$ and $q \geq 1$. Here we are just using the fixed multiplier a rather than the full ideal J appearing in [15]. Note also that in the proofs of the results from [15], the modules F_q are assumed to be free, but the flatness of the modules F_q suffices, nor do we need R to be an integral domain. Then, as in the proof of [15], Theorem 3.5, one uses Lemma 2.4(b) from [8] to show that for $u \in I^{(hn+en)}$, and $q = an + r$, with $0 \leq r < n$, there exists $s \geq 1$ such that

$$I^{s+(h+e)(n-1)}u^a \subseteq (I^{(e+1)})^{[q]}R_S \cap R.$$

Here, S denotes the complement of the union of the associated primes of I. Note that in [15], this latter relation is applied with d, the dimension of R instead of h, but Lemma 2.4(b) in [8] is actually stated with h. The rest of the proof now proceeds exactly as in the proof of [15], Theorem 3.5, showing that $a^n u$ is in the tight closure of $(I^{(e+1)})^n$, as required. The second statement follows from the first statement and the definition of test element. □

The conclusion of the next proposition is a special case of an interesting theorem due to Swanson, namely [20], Theorem 3.1. However, because we only need a special case, the result already follows from [11], Proposition 2.2 and its proof under conditions much more general than Swanson's result.

Proposition 3.4. *Let R be a Noetherian ring and $x \in R$ a non-zerodivisor. Suppose the pair $(x) \subseteq R$ has uniform Artin-Rees number l, i.e., for all ideals $I \subseteq R$, and all $n \geq l$, $I^n \cap (x) \subseteq I^{n-l}(x)$. Then for all ideals $I \subseteq R$, all $n \geq 1$, and all $m \geq nl$, $I^m \cap (x^n) \subseteq I^{m-nl}(x^n)$.*

Proof. Since the conclusion of the proposition holds for $n = 1$ by assumption, we fix $n \geq 2$. For $2 \leq i \leq n$, $(x^{i-1})/(x^i)$ is isomorphic to $R/(x)$, so the pair of modules $(x^{i+1}) \subseteq (x^i)$ has uniform Artin-Rees number l. Since

$$(x^n) \subseteq (x^{n-1}) \subseteq \cdots \subseteq R$$

consists of n containments, it follows from the proof of [11], Proposition 2.2, that the pair $(x^n) \subseteq R$ has uniform Artin-Rees number nl. In other words, for all ideals $I \subseteq R$, all $n \geq 1$ and all $m \geq nl$, $I^m \cap (x^n) \subseteq I^{m-nl}(x^n)$, which is what we want. □

Here is our bootstrapping theorem.

Theorem 3.5. *Let R be a Noetherian ring. Let U be a set of ideals of R, and suppose $x \in R$ is a uniform symbolic multiplier with index $k \geq 1$ for the set U. Assume further that the pair $(x) \subseteq R$ has uniform Artin-Rees number $l \geq 1$. If there exists $b \geq 1$ such that $I^{(b+1)} \subseteq I^{l+1}$, for all ideals $I \in U$, then for $d = k + b$, $I^{(dn)} \subseteq I^n$, for all $n \geq 1$ and all $I \in U$.*

Proof. Let $I \in U$. From our assumption, taking $e = b$ in Definition 3.1, we have

$$x^n I^{(dn)} = x^n I^{(kn+bn)} \subseteq (I^{(b+1)})^n \cap (x^n),$$

for all n. Since $I^{(b+1)} \subseteq I^{l+1}$, it follows that

$$x^n I^{(dn)} \subseteq I^{nl+n} \cap (x^n) \subseteq x^n I^{nl+n-ln} = x^n I^n,$$

where the second containment follows from Proposition 3.4. Cancelling x^n gives $I^{(dn)} \subseteq I^n$, for all $I \in U$ and all n, as required. $\square$

The following corollary is an immediate consequence of the theorem and remark above.

Corollary 3.6. *Let R be a Noetherian domain which is an affine domain over a field of characteristic zero. Then there exists $d \geq 1$ such that R satisfies the uniform symbolic topology property if there exists $c \geq 1$ such that $Q^{(c)} \subseteq Q^d$, for all prime ideals Q.*

4. Simple radical extensions

In this section we study the uniform symbolic topology property in the ring $R = S[\sqrt[n]{a}]$, where S is an integrally closed Noetherain domain satisfying the uniform symbolic topology property, n is a unit in S, and $a \in S$ is square-free. We say that an element $a \in S$ is *square-free* if a is a unit or $aS = \sqrt{aS}$. Equivalently, $a \in S$ is square-free if a is a unit or $QS_Q = aS_Q$, for all height one primes containing a. Note that if $a \in S$ is a non-unit and square-free, then standard field theory implies that $X^n - a$ is irreducible over S, and moreover R is also integrally closed. If a is a unit, then R is integrally closed and even regular, if S is regular, since $n \cdot (\sqrt[n]{a})^{n-1}$ is a unit in R.

To see the potential pitfalls, even in this case of a simple radical extension, consider the case $n = 2$. Thus, $R = S[X]/(X^2 - a)$. Let $Q \subseteq R$ be a prime ideal and set $q = Q \cap S$. Assume $a \notin q$. If $Q \neq qR$, then the simplest case is when Q has the form $(x - b, q)R$, where $b \in S$, $x = \sqrt{a}$ is the residue class of X in R, and $b^2 - a \in q$. This occurs, for example, when S/q is integrally closed. Choose k such that $b^2 - a \in q^{(k)}$. In this case, we claim that for the prime $Q := (x - b, q)R$, we have that $x - b \in Q^{(k)}$ but $x - b \notin Q^2$. To see this notice that in R,

$$(x - b)(x + b) = a - b^2 \in q^{(k)} \subset Q^{(k)},$$

but $x + b \notin Q$, since $a \notin q$. Thus, $x - b \in Q^{(k)}$ as claimed. On the other hand, it is clear that $x - b \notin Q^2$. Thus, for this prime ideal Q, $Q^{(k)} \not\subseteq Q^2$.

Now suppose that for infinitely many values of k there exists $b_k \in R$ and a prime ideal q_k in R such that $b_k^2 - a \in q_k^{(k)}$ and $a \notin q_k$. Then for each such k, $Q_k := (x - b_k, q_k)R$ is a prime ideal satisfying $Q^{(k)} \not\subseteq Q^2$. It follows that R could not have the uniform symbolic topology property. Why can't this happen? When, for example, $(S, \mathfrak{m})$ is a complete local domain, the reason is connected with the strong Artin Approximation theorem. To see this, let b_k and q_k be as above with $b_k^2 - a \in q_k^{(k)}$. By [15], Theorem 2.3, there exists $c \geq 1$ such that $q^{(cn)} \subseteq \mathfrak{m}^n$ for all prime ideals $q \subseteq S$ and all $n \geq 1$, since in a complete local domain, the q-symbolic topology is finer than the $\mathfrak{m}$-adic topology (see also [2], Corollary 2.12). For $k \geq c$, write $k = t_k c + r_k$, where $0 \leq r_k < c$. Then $q_k^{(k)} \subseteq q_k^{(t_k c)} \subseteq \mathfrak{m}^{t_k}$. Thus, $b_k^2 - a \in \mathfrak{m}^{t_k}$, for $k \geq c$. Since $t_k \to \infty$ as $k \to \infty$, one would have approximate solutions of $Y^2 - a = 0$ modulo $\mathfrak{m}^{t_k}$ for infinitely many k. The strong Artin Approximation theorem then gives the existence of an actual solution approximating a given one to a high power of $\mathfrak{m}$. However, since a is assumed to be square-free, this is impossible. For related results and interesting examples, see the paper of Rond [18].

The following theorem is crucial for our main results concerning radical extensions and involves tracking norms of elements from R. Notice that the conclusion of the theorem forces the prime ideal Q to be contained in the radical of $(q, a)R$ - which seems unlikely for an extension that is not a radical extension. Thus, it is not clear how to extend this result directly to more general hypersurfaces.

Theorem 4.1. *Let S be an integrally closed equicharacteristic Noetherian domain satisfying our standard hypothesis and assume that a is a square-free element of S. Assume that S satisfies the uniform symbolic topology property. Let $R = S[X]/(X^n - a)$, and take $Q \in \operatorname{Spec}(R)$ such that $a \notin Q$. Further assume that if char$(R) = p > 0$, then p does not divide n. Set $q = Q \cap S$. Then there is a uniform N, not depending on Q, such that for all $w \geq 1$, $Q^{(Nw)} \subset (aQ^{(w)}, q^w)R$.*

Proof. If a is a unit, the result holds trivially by taking $N = 1$. Otherwise, let $u = b_1x^{n-1} + b_2x^{n-2} + \ldots + b_n \in Q^{(t(n+1))}$ for large t determined below. The ring R is a free S-module with basis the powers of x up to $n-1$. Letting u act on R via multiplication and writing the matrix of the action of u on the basis $\{1, x, \ldots, x^{n-1}\}$ yields the matrix M:

$$\begin{pmatrix} b_n & b_1a & b_2a & \cdots & b_{n-1}a \\ b_{n-1} & b_n & b_1a & \cdots & b_{n-2}a \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ b_2 & b_3 & \cdots & & b_1a \\ b_1 & b_2 & \cdots & \cdots & b_n \end{pmatrix}$$

Let D be the determinant of this matrix. Then $D \in Q^{(t(n+1))} \cap S \subset q^{(t)}$ by [7], Proposition 3.3. Fix m to be the maximum of the uniform Artin-Rees number for the pair $(a) \subset S$ and the uniform Briançon Skoda number for the reduced ring $B = S/(a)$. Note that B is reduced since a is square-free. In particular, for all ideals I of S and $e > m$, $I^e \cap (a) \subset aI^{e-m}$ and for every ideal $J \subset B$, the integral closure of J^e is contained in J^{e-m}. Such a choice is possible since S satisfies the uniform Artin-Rees theorem and B satisfies the uniform Briançon-Skoda theorem.

For each $l \geq 1$, set $N_l = n^l + n^{l-1} + \cdots + n$, and choose $t = 2N_{n+1}mkw$, where k is chosen so that for all prime ideals P in S and for all r, $P^{(rk)} \subset P^r$. This number is independent of P. We now show $N = (n+1)2N_{n+1}mk$ is the N we seek. Thus, if $u \in Q^{(Nw)}$, then

$$D \in q^{(2N_{n+1}mkw)} \subset q^{2N_{n+1}mw}.$$

We claim by induction on $n - i$, starting with $i = 0$, that $b_{n-i} \equiv ac_{n-i}$ modulo $q^{2N_{n-i}mw}$, for some $c_{n-i} \in S$.

Consider $i = 0$. The matrix M becomes upper triangular with b_n along the diagonal when we go modulo the ideal (a). We've seen that $D \in q^{2N_{n+1}mw}$. Working in the reduced ring B it follows that b_n is in the integral closure of $q^{2mw(n^n+\ldots+1)}B$ which is contained in $q^{2mw(n^n+\ldots+1)-m}B$ by our choice of m. Since

$$q^{2mw(n^n+\ldots+1)-m} \subset q^{2mwN_n},$$

we may then write $b_n \equiv ac_n$ modulo q^{2mwN_n} as claimed.

Assume that we have proved the claim up to $i-1$, where $i \leq n-1$. Hence we have that $b_{n-j} \equiv ac_{n-j}$ modulo $q^{2N_{n-i+1}mw}$ for $0 \leq j \leq i-1$.

Let M_i be the matrix whose last $n-i$ rows are the same as that of M, but whose first i-rows are

$$\begin{pmatrix} c_n & b_1 & b_2 & \dots & \dots & \dots & b_{n-1} \\ c_{n-1} & c_n & b_1 & \dots & \dots & \dots & b_{n-2} \\ \vdots & \vdots & \ddots & \ddots & & & \vdots \\ c_{n-i+1} & c_{n-i+2} & \dots & c_n & b_1 & \dots & b_{n-i} \end{pmatrix}$$

By multiplying each of the first i rows of M_i by a, one obtains a matrix that is congruent to M modulo $q^{2N_{n-i+1}mw}$. It follows that

$$D \equiv a^i det(M_i) \text{ modulo } q^{2N_{n-i+1}mw}.$$

Hence,

$$a^i det(M_i) \in q^{2N_{n-i+1}mw}.$$

We may cancel one a at a time by using the fact that for large g, $q^g \cap (a) \subset aq^{g-m}$ by our choice of m. Successively canceling the a, we obtain that

$$det(M_i) \in q^{2N_{n-i+1}mw-mi}.$$

Now consider M_i modulo the ideal (a). Since modulo $q^{2N_{n-i+1}mw}$, $b_n, \dots, b_{n-i+1}$ are divisible by a, over $C := B/q^{2N_{n-i+1}mw-mi}B$, M_i is congruent to the matrix

$$\begin{pmatrix} c_n & b_1 & b_2 & \cdots & b_{n-i} & 0 & \dots & 0 \\ c_{n-1} & c_n & b_1 & \cdots & \cdots & b_{n-i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & & & \ddots & \vdots \\ c_{n-i+1} & c_{n-i+2} & \cdots & c_n & b_1 & \cdots & \cdots & b_{n-i} \\ b_{n-i} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ b_{n-i+1} & b_{n-i} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \cdots & & \vdots \\ b_1 & b_2 & \cdots & b_{n-i} & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Thus, $0 = det(M_i)C = \pm b_{n-i}^n C$. Working in B, we have

$$b_{n-i}^n \in (q^{2(n^{n-i}+\dots+1)mw-m})^n B.$$

(since $i \leq n$). It follows that b_{n-i} is in the integral closure of $q^{2mw(n^{n-i}+\dots+1)-m}B$, which by our assumption on m, belongs to

$$q^{2mw(n^{n-i}+\dots+1)-2m}B \subseteq q^{2mwN_{n-i}}B.$$

We may then write $b_{n-i} \equiv ac_{n-i}$ modulo $q^{2mwN_{n-i}}$ as claimed.

We have now shown that all b_j are divisible by a modulo q^w (in fact, modulo q^{2nmw}), for all $w \geq 1$. It follows that $u = av + r$, where $r \in q^w \subset Q^w$. Then $av \in Q^{(w)}$, and since $a \notin Q$, it follows that $v \in Q^{(w)}$, finishing the proof. $\square$

We next prove that for S regular and $R := S[X]/(X^n - a)$, a^2 is a uniform multiplier for symbolic powers for R with index equal to the dimension of R. The point is to prove the result first in characteristic $p > 0$, and then use reduction to characteristic p when R contains a field of characteristic zero.

Proposition 4.2. *Let S be an F-finite regular domain containing a field of characteristic $p > 0$ and assume $\dim(S) = d$. Let $a \in S$ be square-free and suppose $n \geq 2$ is not divisible by p. Then for $R := S[\sqrt[n]{a}]$, $aR^{1/q} \subseteq S^{1/q}[R]$, for all $q \geq 1$. In particular, a^2 is a uniform multiplier for symbolic powers for all ideals, with index d.*

Proof. We first note that since S is regular and R is free over S, each $F_q := S^{1/q} \otimes_S R = S^{1/q}[R]$ is flat over R, since each $S^{1/q}$ is finite and flat over S and R is torsion-free over S. Note also that R inherits the F-finite property from S. To prove the first statement in the proposition, note that $S^{1/q}[R] = S^{1/q}[\sqrt[n]{a}]$ and $R^{1/q} = S^{1/q}[\sqrt[nq]{a}]$. For fixed $q \geq 1$ and any $r \geq 1$, we can find positive integers $\alpha, \beta \geq 1$ such that $r + nq = \alpha n + \beta q$. Thus, $\frac{r}{nq} + 1 = \alpha(\frac{1}{q}) + \beta(\frac{1}{n})$. It follows that

$$a \cdot (a^{\frac{1}{nq}})^r = (a^{\frac{1}{q}})^\alpha \cdot (a^{\frac{1}{n}})^\beta \in S^{1/q}[R].$$

Since this holds for all $r \geq 1$, this gives what we want. For the second statement, if a is a non-unit, by Lemma 3.3, it suffices to observe that a is a test element for R - but this follows from [9], Theorem 6.9. If a is a unit, a^2 is a uniform symbolic multiplier since R is regular. $\square$

Proposition 4.3. *Let S be an excellent regular domain containing a field and suppose R is a finite extension of the form $R = S[\sqrt[n]{a}]$, where a is square-free. Further assume that if S has characteristic $p > 0$, then S is F-finite and p does not divide n. Then a^2 is a uniform symbolic multipler with index $d := \dim(R)$ for the set of radical ideals in R.*

Proof. We may assume that a is not a unit. The case that S contains a field of characteristic p follows from Proposition 4.2.

If S contains a field of characteristic zero, the proof of the result we seek proceeds via reduction to characteristic p. The proof follows along the same lines as most reduction to characteristic p proofs. In particular, we can follow the ideas in the proofs of [8], Theorems 4.3 and 4.4, and also the proof of Theorem 4.7 and the proof in the Appendix of [13]. To elaborate, the results in [8], show how, starting with a complete local ring A, say, and a counter-example to an inclusion of the type we want involving symbolic powers, one can produce a counter-example in a ring of positive characteristic - the point being that conditions like elements belonging to, or not belonging to, various symbolic powers, as well as the maximum of local analytic spreads can be preserved via the reduction process. On the other hand, we need a slight variation of this, because we will be working with two rings at once, S and its simple extension R – but [13], Theorem 4.7 and the Appendix, illustrate how to carry the ring strucure of R along in the reduction process. Another crucial point here is that the failure of the required property of a proposed uniform multiplier for symbolic powers can be preserved along the way, because the element a^2 is given *a priori* as an element of the original ring – in this case R.

We now sketch out the steps required in order to reduce our statement to positive characteristic. Set $x = \sqrt[n]{a}$ and assume that we have a reduced ideal $I \subseteq R$, $u \in I^{(hn+en)}$ with $a^{2n}u \notin (I^{(e+1)})^n$, for some $e \geq 0$, $n \geq 1$. By standard localization arguments, our counter-example persists after we localize at some prime ideal in S, so we may assume that we have a counter-example when S is a regular local ring. We now lift the counterexample by completing S at its maximal ideal. Writing $\hat{S}$ for the completion of S, we have $\hat{R} = \hat{S} \otimes_S R$, which is faithfully flat over R.

Note also that since S is excellent, $\hat{R}$ remains reduced, as do $\hat{R}/I\hat{R} = \hat{R}/\hat{I}$ and $\hat{R}/a\hat{R}$. Moreover, since $\hat{R}/I\hat{R}$ is faithfully flat over R/I, non-zerodivisors on the latter remain non-zerodivisors on the former, so that if $U \subseteq R$ denotes the set of non-zerodivisors on R/I and $W \subseteq \hat{R}$ denotes the set of non-zerodivisors on $\hat{R}/\hat{I}$, then $U = W \cap R$ and hence $(R/I^k)_U \hookrightarrow (\hat{R}/\hat{I}^k)_W$, so that $\hat{I}^{(k)} \cap R = I^{(k)}$, for all $k \geq 1$. Thus, $u \in \hat{I}^{(hn+en)}$. On the other hand, we clearly have

$$I^{(k)}\hat{R} = I^k\hat{R}_U \cap \hat{R} \subseteq I^k\hat{R}_W \cap \hat{R} = \hat{I}^{(k)}.$$

If P is an associated prime of $\hat{R}/\hat{I}^{(k)}$, $P \cap W = \emptyset$, so P is contained in a minimal prime of $\hat{I}$, and thus P is also an associated prime of $I^{(k)}\hat{R}$. It follows from this that $I^{(k)}\hat{R} = \hat{I}^{(k)}$, for all k. Therefore, $((I\hat{R})^{(e+1)})^n = (\hat{I}^{(e+1)})^n$, and therefore, $a^n u \notin (\hat{I}^{(e+1)})^n$. So, we may begin again assuming that S is a complete regular local ring, and we have a counter-example in R as above to our proposition.

At this point one uses Artin approximation to find an counter-example in an affine algebra over a field of characteristic zero. If we were only working with S, then by [8], Theorem 4.3 we could create a counter-example in an affine algebra, but we need to preserve our counter-example in a ring over S. We may therefore, follow the path laid out in [13], Theorem 4.7 and the Appendix. One uses equations over S to capture the ring structure of R. For example, since R is free over S with basis $1, x, \ldots, x^{n-1}$, where $f(X)$ has degree n, one writes each product $x^i \cdot x^j$ in terms of the basis with coefficients in S. The resulting equations can be thought of solutions over S to a system of equations in n variables over S. Similarly, one can realize the associative property of multiplication and the distributive property as solutions to equations over S. Since the ideal I is a submodule of R as an S-module, one can choose a set of generators for I and write equations expressing the closure of I under multiplication by elements of R, using the consequences of taking products of the basis elements of R over S with the generators of I as an S-module. As in [8], Theorem 4.3, one can transfer all of this data and the attendant data associated to our counter-example to a finitely generated algebra over the coefficient field, say E, of S. Here we are thinking of S as a formal power series ring in d variables over E. In fact, one first adjoins to E all of the relevant elements from S that are solutions to the various equations tracking the data to obtain a subring ring S_0 and then uses [8] (which relies upon [3]) to find a ring S_1 and maps $S_0 \hookrightarrow S_1 \to S$ in which all of the conditions from S are preserved, and such that all of the ideals and modules that we started with in S are obtained by tensoring their counterparts in S_1 with S over S_1. Moreover, S_1 is a regular ring and the counter-example in question holds in the extension $R_1 := S_1[x]$. Note that this can be done so that the element x still satisfies the equation $x^n - a = 0$, a is square-free in S_1 and the rings R_1 and R_1/I_1 are reduced, where I_1 is generated by the images in S_1 of the original generators of I. Now, strictly speaking, the field E is not the original field E_0 contained in the original S, but one can assume $E_0 \subseteq E$, and the last paragraph of [10], Theorem 3.5.1 explains how to reduce to the case that $E_0 = E$.

The next step is to reduce to an affine algebra over $\mathbb{Z}$, which can be done in a standard way by collecting all coefficients of all the finitely many equations which describe our situation, and then letting A be the finitely generated $\mathbb{Z}$-algebra obtained as the subring of the base field given by adjoining those finitely many elements to $\mathbb{Z}$. One further uses generic flatness to insure that after creating models R_A and $(R/I)_A$ of R and R/I over this finitely generated $\mathbb{Z}$-subalgebra A of k, there exists a dense subset of closed points $S \subseteq \operatorname{Spec} A$ such that we still a counterexample after moding out any one of the closed points in S. These counterexamples now live in positive characteristic, and by choosing the characteristic large enough we can avoid any divisors of our fixed integer n. Moreover, as described in Chapter 2 of [10] and [8, Theorem 4.3], we retain all relevant information, including analytic spreads and the various ideals being reduced. This leads to a contradiction by Proposition 4.2. $\square$

The following is the main result of this section.

Theorem 4.4. *Let S be a finite dimensional excellent regular domain containing a field and suppose that S satisfies our standard hypothesis. Take $a \in S$ square-free, $n \geq 1$ and set $R := S[\sqrt[n]{a}]$. If S has positive characteristic, assume that S is F-finite, and p does not divide n. Then $R := S[\sqrt[n]{a}]$ satisfies the uniform symbolic topology property on prime ideals.*

Proof. By Proposition 4.3, a^2 is a uniform multiplier for symbolic powers for R with index d. In particular, $a^{2n}Q^{(dn)} \subseteq Q^n$ for all n, where $d := \dim(S)$.

Now, consider the set U of prime ideals in R not containing a. We claim there exists $k_1 \geq 1$ such that $Q^{(k_1 n)} \subseteq Q^n$, for all $Q \in U$ and $n \geq 1$. Since in either characteristic, R admits a uniform multiplier for symbolic powers with respect to U (namely a^2), by Theorem 3.5 it suffices to find $b \geq 1$ such that for all $Q \in U$, $Q^{(b+1)} \subseteq Q^{l+1}$, where l is the uniform Artin-Rees number for $(a^2) \subseteq R$. For any such Q, set $q = Q \cap S$. Then by Theorem 4.1, there exists $N \geq 1$, independent of Q, such that $Q^{(Nw)} \subseteq (aQ^{(w)}, q^w)$, for all $w \geq 1$. It follows from this, that for any $t \geq 1$, $Q^{(N^t w)} \subseteq (a^t Q^{(w)}, q^w)$, for all $w \geq 1$.

Thus, taking $t = 2(l+1)$ and $w = (l+1)d$, we have for all $Q \in U$:

$$Q^{(N^{2(l+1)}(l+1)d)} \subseteq (a^{2(l+1)}Q^{((l+1)d)}, q^{d(l+1)})) \subseteq (Q^{l+1}, q^{d(l+1)}) = Q^{l+1}.$$

Thus, if we set $b = N^{2(l+1)}(l+1)d - 1$, we obtain $Q^{(b+1)} \subseteq Q^{l+1}$, for all Q not containing a, which gives the k_1 we seek.

Now, let V denote the set of prime ideals in R containing a. Then for any $Q \in V$, Q is the only prime in R lying over $Q \cap S$. Thus, by Theorem 2.4, there exists $k_2 \geq 1$, such that $Q^{(k_2 n)} \subseteq Q^n$, for all $Q \in V$ and $n \geq 1$. If we take $c = \max\{k_1, k_2\}$, it follows that $Q^{(cn)} \subseteq Q^n$ for all Q in $\operatorname{Spec}(R)$ and $n \geq 1$, which is what we want. $\square$

Remark 4.5. Unfortunately, we are not able to extend Theorem 4.4 to the case of radical ideals. If $I \subseteq S[\sqrt[n]{a}]$ is a radical ideal and $a \in S$ is a square-free element not belonging to any minimal prime of I, then the proof of Theorem 4.1 goes through, and thus also, the corresponding part of Theorem 4.4. On the other hand, if a belongs to every minimal prime of I, then by making a minor modification of the Ascent Theorem, one can show that the corresponding part of Theorem 4.4 also goes through. The problem comes when I is an intersection of both types of primes. In this case we can write $I = K \cap L$, where every minimal prime over K does not contain a and every minimal prime over L contains a. There is a uniform c such that $K^{(cn)} \subseteq K^n$ and $L^{(cn)} \subseteq L^n$, for all such K, L and all n. Thus, $I^{(cn)} \subseteq K^n \cap L^n$. It is easy to show there exists $t \geq 1$ such that $K^{nt} \cap L^{nt} \subseteq I^n$ and thus $I^{(ctn)} \subseteq I^n$ for all n, but we do not know if such a t exists uniformly, independent of I.

5. Repeated radical extensions

In this section we extend the main result in the previous section to the case $R = S[\sqrt[n]{a_1}, \dots, \sqrt[n]{a_r}]$, with $a_1, \dots, a_t \in S$ square-free and sufficiently general.

We begin with a definition.

Definition 5.1. Let S be an integrally closed domain, $n \geq 1$, and $a_1, \dots, a_r$ square-free elements. We say that n and $a_1, \dots, a_r$ satisfy property (*) if n is a unit and no two of a_i and a_j are contained in the same height one prime of S.

Proposition 5.2. *Let S be an integrally closed Noetherian domain, $n \geq 1$ and $a_1, \dots a_r \in S$ square-free elements satisfying property (*). Then $a_2, \dots, a_r$ are square-free in $S[\sqrt[n]{a_1}]$.*

Proof. Fix $b := a_i$, $i \neq 1$ and set $a_1 = a$. If b is not a unit we have to show that if $Q \subseteq S[\sqrt[n]{a}]$ is a height one prime containing b, then $bS[\sqrt[n]{a}]_Q = QS[\sqrt[n]{a}]_Q$. Fix such a Q and let $q = Q \cap S$. For $f(x) = X^n - a \in S[X]$, consider its image in $k(q)[X]$. By our assumption on n and the height one primes containing a and b, the images of $f'(X)$ and $f(X)$ are relatively prime over $k(q)$, so $f(X)$ has distinct roots over the algebraic closure of $k(q)$, and thus $f(X)$ factors as a product of distinct irreducible polynomials over $k(q)$, say $f(X) \equiv p_1(X) \cdots p_r(X)$. It follows that in $S_q[X]$,

$$Q'_q = (q, p_i(X))S_q[X],$$

for some i, where Q' is the lift of Q to $S[X]$. Without loss of generality, we assume $i = 1$. On the other hand, in $S_q[X]$,

$$f(X) = p_1(X) \cdots p_r(X) + q_0(X),$$

where $q_0(X) \in qS_q[X]$. Thus,

$$f(X)S[X]_{Q'} = (p_1(X) + uq_0(X))S[X]_{Q'},$$

for for $u = (p_2(X) \cdots p_r(X))^{-1}$. Thus, in $S[\sqrt[n]{a}]_Q$, $p_1(X) \equiv -uq_0(X)$, and hence $QS[\sqrt[n]{a}]_Q = qS[\sqrt[n]{a}]_Q$. Since $bS_q = qS_q$, we have $bS[\sqrt[n]{a}]_Q = QS[\sqrt[n]{a}]_Q$, which is what we want. $\square$

Proposition 5.3. *Let S be a Noetherian integrally closed domain, $n \geq 1$, and $a_1, \dots, a_r$ square-free elements satisfying (*). Then $R = S[\sqrt[n]{a_1}, \dots, \sqrt[n]{a_r}]$ is integrally closed.*

Proof. Induct on r. If $r = 1$, then as mentioned in the first paragraph of the previous section, R is integrally closed. Suppose $r > 1$. Set $T := S[\sqrt[n]{a_1}]$. Then by the Proposition 5.2, $a_2, \dots, a_r$ are square-free in T. Morevover, it is clear that for $i, j \geq 2$, no two of a_i and a_j are contained in the same height one prime of T. Thus, $R = T[\sqrt[n]{a_2}, \dots, \sqrt[n]{a_r}]$ is integrally closed, which gives what we want. $\square$

Proposition 5.4. *Let S be a finite dimensional excellent regular domain containing a field and fix $n \geq 1$. Assume that if S has characteristic $p > 0$, then S is F-finite and p does not divide n. Let $a_1, \dots, a_r$ be square-free elements in S satisfying condition (*). Set $R := S[\sqrt[n]{a_1}, \dots, \sqrt[n]{a_r}]$ and $a := a_1 \cdots a_r$. Then a^2 is a uniform multiplier for symbolic powers with index $d := \dim(R)$ for the set of radical ideals in R.*

Proof. The proof is similar to the proof of Proposition 4.3. We begin by considering the case that S has characteristic $p > 0$. For each $1 \leq i \leq r$, set $R_i := S[\sqrt[n]{a_i}]$. By Proposition 4.2, $a_i R_i^{1/q} \subseteq S^{1/q}[R_i]$, for all i and all q. If we set $a := a_1 \cdots a_r$, it follows that $aR^{1/q} \subseteq S^{1/q}[R]$, for all q. As in the proof of Proposition 4.2,

$$F_q = S^{1/q} \otimes_S R = S^{1/q}[R]$$

is a flat R-module for all q, since $S^{1/q}$ is a flat S-module. On the other hand, a is a test element by [9], Theorem 6.9, and so this case follows from Lemma 3.3. If S has characteristic zero, we may reduce to the case of positive characteristic, as in the proof of Proposition 4.3. □

Theorem 5.5. *Let S be a finite dimensional regular domain containing a field and assume S satisfies our standing hypothesis. Fix $n \geq 1$ and assume that if S has characteristic $p > 0$, then S is F-finite and p does not divide n. Let $a_1, \ldots, a_r$ be square-free elements in S satisfying condition (*). Then $R := S[\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_r}]$ satisfies the uniform symbolic topology property on prime ideals.*

Proof. Again, the proof is similar to the case of a simple radical extension. We proceed by induction on r, the case $r = 1$ having been handled in the previous section. Set $a := a_1 \cdots a_r$, and for each $1 \leq i \leq r$, set

$$R_i := S[\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_{i-1}}, \sqrt[n]{a_{i+1}}, \ldots, \sqrt[n]{a_r}].$$

Each R_i satisfies the uniform symbolic topology property, by our induction hypothesis.

Let us first consider the set of prime ideals $U \subseteq \mathrm{Spec}(R)$ not containing a. Then, by Proposition 5.4, a^2 is a uniform multiplier for symbolic powers for U with index d. On the other hand, by Propositions 5.2 and 5.3, each R_i is integrally closed and a_i is square-free in R_i. Thus, by Theorem 4.1, there exists N_i such that for all $Q \in U$,

$$Q^{(N_i w)} \subseteq (a_i Q^{(w)}, q_i^w),$$

for all $w \geq 1$, where $q_i = Q \cap R_i$. Thus,

$$Q^{(N_i w)} \subseteq (a_i Q^{(w)}, Q^w),$$

for all w. It follows from this, that if we set $N := N_1 \cdots N_r$, then

$$Q^{(Nw)} \subseteq (aQ^{(w)}, Q^w),$$

for all $w \geq 1$ and $Q \in U$, and thus,

$$Q^{(N^t w)} \subseteq (a^t Q^{(w)}, Q^w),$$

for all $t \geq 1$, $w \geq 1$ and $Q \in U$. We are now in the same situation as in the proof of Theorem 4.4, so that if we let l denote the uniform Artin-Rees number for the pair $(a^2) \subseteq R$, $t := (l+1)2$, $w := (l+1)d$, and $b = N^{2(l+1)}(l+1)d - 1$, then $Q^{(b+1)} \subseteq Q^{l+1}$, for all $Q \in U$. Since a^2 is a uniform multiplier for symbolic powers for U with index d, it follows from Theorem 3.5, that there exists $k \geq 1$ such that $Q^{(kn)} \subseteq Q^n$, for all $n \geq 1$ and all $Q \in U$.

We now consider the set of primes $V \subseteq \mathrm{Spec}(R)$ containing a. We can write $V = V_1 \cup \cdots \cup V_r$, where V_i denotes the set of primes in R containing a_i. For each prime $Q \in V_i$, Q is the only prime in R lying over $Q \cap R_i$. Thus, by Theorem 2.4 and our induction hypothesis, there exists c_i such that $Q^{(c_i n)} \subseteq Q^n$, for all $Q \in V_i$

and $n \geq 1$. If we now take $c = \max\{k, c_1, \dots, c_r\}$, it follows that $Q^{(cn)} \subseteq Q^n$, for all $n \geq 1$ and all $Q \subseteq R$. $\square$

6. Abelian extensions

We begin by recalling the basic fact of Kummer theory. Let $L \subseteq K$ be a finite, Galois extension of fields such that the Galois group of K over L is abelian. Let n denote the index of the Galois group of K over L, i.e., n is the least positive integer such that n^{th} power of every element in the group is the identity element. Then Kummer theory states that if L contains a primitive n^{th} root of unity, then there exist $c_1, \dots, c_s \in L$ such that $K = L(\sqrt[n]{c_1}, \dots, \sqrt[n]{c_s})$. Here, one must assume that if L has positive characteristic $p > 0$, then p does not divide n.

The next theorem is the main theorem of our paper. Most of the hard work has been done. What remains is to reduce to the case of a repeated radical extension.

Theorem 6.1. *Let S be a finite dimensional excellent regular domain containing a field and suppose that S satisfies our standing hypothesis. Let R be an abelian extension of S and assume that if S has characteristic $p > 0$, then S is F-finite and p does not divide the index of the associated Galois group. Then R satisfies the uniform symbolic topology property on prime ideals.*

Proof. Let L denote the quotient field of S and K the quotient field of R. Thus K is a Galois extension of L with abelian Galois group and R is the integral closure of S in K. Let n be the index of the Galois group of K over L. Since the uniform symbolic topology property descends in a finite extension of integrally closed domains, by Theorem 2.5, it suffices to show R is contained in an extension of the form $S[\sqrt[n]{a_1}, \dots, \sqrt[n]{a_r}]$, such that the conditions in Theorem 5.5 hold. In fact, we will have to make a few minor modifications to S and R to obtain such a relation, but the strategy is still the same.

Let ϵ denote a primitive n^{th} root of unity. We will first reduce to the case that $\epsilon \in L$, or equivalently, $\epsilon \in S$. Suppose $\epsilon \notin L$. Then the Galois group of $K(\epsilon)$ over $L(\epsilon)$ is a subgroup of the Galois group of K over L (even if $\epsilon \in K$), thus the former group is abelian, and its index divides n, and thus is not divisible by p, if the characteristic of L is p. Set $S_1 := S[\epsilon]$, so that the quotient field of S_1 is $L(\epsilon)$ and S_1 is a regular domain satisfying the hypotheses of the theorem. The proof that S_1 is regular is similar to the proof in Proposition 5.2. In other words, it suffices to see that if $g(X)$ is the minimal polynomial of ϵ over L, then for any prime $q \subseteq S$, $g(X)$ factors as a product of distinct irreducible polynomials over $k(q)$. But since the images of $X^n - 1$ and its derivative are relatively prime over $k(q)$, $X^n - 1$ factors as a product of distinct irreducibles over $k(q)$, so the same applies to $g(X)$, and thus S_1 is regular. Now if R_1 denotes the integral closure of S_1 in $K(\epsilon)$ and the conclusion of the theorem holds for R_1, then since the uniform symbolic topology property descends in the finite extension $R \subseteq R_1$, R satisfies the uniform symbolic topology property. Thus, we may begin again, assuming that S contains a primitive n^{th} root of unity.

We now reduce to the case where S is a UFD. Let X be an indeterminate and write $S(X)$ for the ring $S[X]$ localized at the set of polynomials whose coefficients generate S. Note also that $S(X)$ is a regular domain satisfying the hypotheses of the theorem, since the maximal ideals of $S(X)$ are of the form $mS(X)$, for $m \subseteq S$ a maximal ideal. Thus, $S(X)$ is locally a UFD. On the other hand, since invertible ideals in $S(X)$ are principal ([1], Theorem 2.1 (5); see also [6]), $S(X)$ is a UFD. It is straight forward to check that $S(X) \subseteq R(X)$ is an abelian extension with the same Galois group as $S \subseteq R$. Suppose there exists $c \geq 1$ such that $P^{(cn)} \subseteq P^n$, for all prime ideals $P \subseteq R(X)$ and all $n \geq 1$. For any prime ideal $Q \subseteq R$, $QR(X)$ is a prime ideal satisfying

$$Q^{(cn)}R(X) = (QR(X))^{(cn)} \quad \text{and} \quad Q^nR(X) \cap R = Q^n,$$

for all n. It follows that $Q^{(cn)} \subseteq Q^n$, for all $Q \subseteq R$ and all n. Thus, we may begin again assuming that, in addition to our hypotheses on S, S is a UFD containing a primitive nth root of unity.

By Kummer theory, there exist $c_1, \ldots, c_s$ such that $K = L(\sqrt[n]{c_1}, \ldots, \sqrt[n]{c_s})$, with each $c_j \in L$. Suppose we could find $a_1, \ldots a_r$ square-free elements in S satisfying (*) such that $K \subseteq K' := L(\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_r})$. If T is the integral closure of R in K', then on the one hand, since the uniform symbolic topology property descends in a finite extension of integrally closed domains, R will have the uniform symbolic topology property if T does. On the other hand, since $S[\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_r}]$ is integrally closed by Proposition 5.3, $T = S[\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_r}]$, and so T has the uniform symbolic topology property, by Theorem 5.5. Thus, it remains to find $a_1, \ldots a_r$ square-free elements in S satisfying (*), so that $K \subseteq K'$. For this, we first note that if $c \in L$ can be written as $c = \frac{a}{b}$, with $a, b \in S$, then $L(\sqrt[n]{c}) \subseteq L(\sqrt[n]{a}, \sqrt[n]{b})$, Thus, we may enlarge K and begin again, assuming $K = L(\sqrt[n]{c_1}, \ldots, \sqrt[n]{c_s})$ with each $c_i \in S$.

Now, since S is a UFD, each c_i is a unit or product of prime elements. Let $a_1, \ldots, a_r$ denote the collection of prime elements or units occurring as a factor of some c_i. These are clearly square-free elements satisfying (*). Suppose that in S we can write each $c_i := a_{i_1}^{d_{i_1}n+e_i} \cdots a_{i_t}^{d_{i_t}n+e_t}$, where each $0 \leq e_i < n$. Then for

$$\gamma_i := a_{i_1}^{d_{i_1}}(\sqrt[n]{a_{i_1}})^{e_{i_1}} \cdots a_{i_t}^{d_{i_t}}(\sqrt[n]{a_{i_t}})^{e_{i_t}},$$

$\gamma^n = c_i$, so $S[\sqrt[n]{c_i}] \subseteq S[\sqrt[n]{a_{i_1}}, \ldots, \sqrt[n]{a_{i_r}}]$, and hence $L(\sqrt[n]{c_i}) \subseteq L(\sqrt[n]{a_{i_1}}, \ldots, \sqrt[n]{a_{i_r}})$. Doing this for each c_i shows K is contained in $L(\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_r})$, which is what we want. $\square$

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