HILBERT FUNCTIONS OF BIGRADED ALGEBRAS

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Section 1. Introduction.

Let S be an Artinian local ring. Let $X = \{X_0, \ldots, X_m\}$ and $Y = \{Y_0, \ldots, Y_n\}$ be two sets of indeterminates. Then the polynomial ring $R = S[X;Y]$ is a bigraded S-algebra. Let R_{rs} denote the S-module generated by monomials of the form PQ where P is a degree r monomial in X and Q is a degree s monomial in Y. We say that PQ is a monomial of degree (r,s). R decomposes as $R = \bigoplus_{r,s\geq 0} R_{rs}$ and $R_{rs}R_{ab} = R_{r+a,s+b}$ for all $r, s, a, b, \in \mathbb{N}$. An element of R_{rs} is called bihomogenous of degree (r,s). An ideal $I \subseteq R$ generated by bihomogeneous elements is called a bihomogeneous ideal. Therefore $A = R/I$ is a bigraded algebra, the bigraded component of degree (r,s) being $A_{rs} = R_{rs}/I_{rs}$. The Hilbert function of A is defined as

$$H(r,s) = \lambda(A_{rs})$$

where λ denotes length as an S-module. Van der Waerden [W] studied the function $H(r,s)$. The original idea of Hilbert functions of multigraded algebras is due to Lasker [L], as pointed out by Van der Waerden in [W].

In [W], it is proved that if S is a field and $d = \dim(A) - 2$, then for large r and s, $H(r,s)$ is given by a polynomial $P(r,s)$ of the form $\sum_{i+j\leq d} a_{ij}\binom{r}{i}\binom{s}{j}$, where a_{ij} are integers. This has been extended to the Artinian case by Bhattacharya [B]. Among the numbers a_{ij}, the ones for which $i+j = d$ are especially interesting. Let us denote them by $e_{ij}(I)$. These are called the *degrees* of I in [W]. Van der Waerden proved that the degrees are non-negative integers and pointed out their geometric interpretation.

A proper bihomogeneous ideal I is called projectively irrelevant if for some non-negative integers a and b, $(X)^a(Y)^b \subseteq I$, projectively relevant otherwise. According to Theorem

Commutative algebra, Aron Simis, N.V. Trung and G. Valla (Eds), Proceedings of "Workshop in Commutative algebra", ICTP, Trieste (1992), World Scientific (1994), 291-302

6 of [B], if I is not projectively irrelevant, then $e_{ij}(I) > 0$ for all i, j. This is clearly not true. For example take $I = 0$ (examples with $I \neq 0$ can be constructed; see section 5). Then $H(r,s) = \lambda(S)\binom{r+m}{m}\binom{s+n}{n}$ for all $r, s \geq 0$. Therefore $e_{mn} = \lambda(S)$ and the other e_{ij} are zero. In this note we point out some of the elementary differences between Hilbert functions of bigraded versus N-graded algebras and indicate a few of the similarities as well. In section two we will identify the total degree of $P(r,s)$. In general, it is not quite the "expected value" $\dim(A) - 2$. In section three, we will prove a formula for the degrees of I using the Hilbert series

$$Q(t_1, t_2; A) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \lambda(A_{rs}) t_1^r t_2^s.$$

The bigraded algebra A can be made into a graded algebra by setting $A_n = \bigoplus_{r+s=n} A_{rs}$. Hence $A = \bigoplus_{n \geq 0} A_n$. In section four we will show that the multiplicity of A, $e(A)$, is the sum of the degrees of I provided the ideals generated by A_{10} and A_{01} have positive height. As a consequence, we recover a formula proved in [V] about the multiplicity of Rees algebras localized at their homogeneous maximal ideals. Finally, in section five we present some examples.

Section 2. The total degree of the Hilbert polynomial.

The results of Bhattacharya and Van der Waerden show that the total degree of the Hilbert polynomial $P(r,s;A)$ corresponding to the Hilbert function $H(r,s) = \lambda(A_{rs})$ is at most $\dim(A) - 2 := d$. To see that the total degree of $P(r,s;A)$ is not d even if I is a projectively relevant ideal, consider the ideal $I = (X_0) \cap (Y_0, Y_1)$ in the bigraded polynomial ring $R = k[X_0; Y_0, Y_1, Y_2]$ over a field k. Then for $A = R/I$, $\dim(A) = 3$ and I is projectively relevant. Set $P = (X_0)$ and $Q = (Y_0, Y_1)$. From the exact sequence

$$0 \to A \to R/P \oplus R/Q \to R/(P+Q) \to 0$$

we obtain $P(r,s;A) = P(r,s;R/P) + P(r,s;R/Q) - P(r,s;R/P+Q) = 0 + 1 - 0 = 1$. Thus $\deg(P(r,s;A)) = 0 < \dim(A) - 2$. In this section we will determine the total degree, $\deg(P(r,s;A))$ of the Hilbert polynomial of A.

Definition 2.1. *The relevant dimension of $A = R/I$, denoted by* $\mathrm{rdim}(A)$ *is defined as* $max\{\dim(R/P)\}$, *where P ranges over the projectively relevant primes associated to I.*

Theorem 2.2. *Let S be an Artinian local ring and $R = S[X;Y]$ as above be the polynomial ring in $m+n+2$ indeterminates. Let $I \subseteq R$ be a relevant bihomogeneous ideal and put $A = R/I$. Let $P(r,s;A)$ denote the Hilbert polynomial corresponding to the Hilbert function $H(r,s) = \lambda(A_{rs})$. Then $\deg(P(r,s;A)) = \operatorname{rdim}(A) - 2$.*

Proof. Since I is relevant, at least one of the primary components of I is relevant. By taking a primary decompostion in which all the primary components are bihomogenous, we can write $I = J \cap K$, where all the associated primes of J (resp. K) are projectively irrelevant (resp. relevant). Then $J + K$ is also projectively irrelevant. Therefore $P(r,s;R/J) = P(r,s;R/J+K) = 0$. By the exact sequence

$$0 \to A \to R/J \oplus R/K \to R/J+K \to 0$$

we get $P(r,s;A) = P(r,s;R/K)$. Since $\operatorname{rdim}(A) = \operatorname{rdim}(R/K) = \dim(R/K)$, we may assume that I has no projectively irrelevant components and we are reduced to proving that $\deg(P(r,s;A)) = \dim(A) - 2$. If L is a bihomogeneous prime with $\dim(R/L) \leq 1$, then L is projectively irrelevant. Hence $\dim(A) \geq 2$. Set $\dim(A) = d+2$. We can view A as a graded ring $\bigoplus_{n\geq 0} A_n$ whose nth graded component A_n is $\bigoplus_{r+s=n} A_{rs}$. Assume $P(r,s;A) = H(r,s)$ for all $r,s \geq q$. For $n \geq 2q$, set $n = 2q+k$. Put

$$B_n = A_{n0} \oplus A_{n-1,1} \oplus \cdots \oplus A_{n-q+1,q-1}$$

$$C_n = A_{n-q,q} \oplus A_{n-q-1,q+1} \oplus \cdots \oplus A_{q,n-q}$$

$$D_n = A_{q-1,n-q+1} \oplus \cdots \oplus A_{0n}.$$

Since I has no projectively irrelevant component, the ideals $(X)A$ and $(Y)A$ have positive height. Set $E_i = \bigoplus_{n\geq i} A_{n-i,i}$ and $F_i = \bigoplus_{n\geq i} A_{i,n-i}$. Then $E_0 \cong A/(Y)A$ and $F_0 \cong A/(X)A$. Therefore $\dim(E_0) \leq d+1$ and $\dim(F_0) \leq d+1$. Since E_i (resp. F_i) is an E_0 (resp. F_0)-module, $\dim(E_i) \leq d+1$ and $\dim(F_i) \leq d+1$. Hence for large n, $\lambda(A_{n-i,i})$ and $\lambda(A_{i,n-i})$ are given by polynomials of degree at most d. Hence $\lambda(B_n)$ and $\lambda(D_n)$ are given by polynomials of degree at most d. Since

$$\lambda(A_n) = \lambda(B_n) + \lambda(C_n) + \lambda(D_n),$$

and since $\lambda(A_n)$ is given by a polynomial of degree $d+1$ for large n, $\lambda(C_n)$ is given by a polynomial of degree $d+1$ for large n. Since $n = 2q+k$, $n-q = q+k$. Hence for $i = 0,1,\ldots,k$,

$$\lambda(A_{n-q-i,q+i}) = P(n-q-i, q+i; A).$$

Thus

$$\lambda(C_n) = \sum_{i=0}^{n-2q} P(n-q-i, q+i; A).$$

Since $\deg(P(r,s;A)) \leq d$, we may now conclude that $\deg(P(r,s;A)) = d$.

Section 3. The Hilbert series.

Let X, Y, S, R, I and A have the same meaning as in the introduction. The Hilbert series of A is given by $Q(t_1,t_2) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \lambda(A_{rs}) t_1^r t_2^s$. By an argument analogous to the proof of Theorem 1.11 of [A-M], there exists a polynomial $N(t_1,t_2) \in \mathbb{Z}[t_1,t_2]$ so that $Q(t_1,t_2) = N(t_1,t_2)/(1-t_1)^{m+1}(1-t_2)^{n+1}$. Write the Hilbert polynomial of A as $P(r,s;A) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}\binom{r+i}{i}\binom{s+j}{j}$.

Theorem 3.1. *For $i = 0,1,\ldots,m$ and $j = 0,1,\ldots,n$:*

$$a_{ij} = \frac{(-1)^{m+n-i-j}}{(m-i)!(n-j)!} \frac{\partial^{m+n-i-j} N}{\partial t_1^{m-i} \partial t_2^{n-j}}\bigg|_{t_1=t_2=1}$$

Proof. The argument is essentially the same as the one for the coefficients of the Hilbert polynomial for N-graded rings. For the sake of completeness, we include the proof. The proof is modeled after the proof of Proposition 4.1.g of [B-H]. We write

$$N^{(i,j)} = \frac{\partial^{i+j} N(t_1,t_2)}{\partial t_1^i \partial t_2^j}\bigg|_{t_1=t_2=1}.$$

Then :

$$Q(t_1,t_2) - \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{N^{(i,j)}(-1)^{i+j}}{i!j!(1-t_1)^{m+1-i}(1-t_2)^{n+1-j}} = \frac{D(t_1,t_2)}{(1-t_1)^{m+1}(1-t_2)^{n+1}}$$

where

$$D(t_1,t_2) = N(t_1,t_2) - \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{N^{(i,j)}}{i!j!}(t_1-1)^i(t_2-1)^j.$$

Hence, $D(t_1,t_2)$ is the remainder of the Taylor series of $N(t_1,t_2)$ about the point $(1,1)$ having terms of degree $\geq m+1$ in $t_1 - 1$ and degree $\geq n+1$ in $t_2 - 1$. Thus $D(t_1,t_2)$ is divisible by $(1-t_1)^{m+1}(1-t_2)^{n+1}$. Therefore $\lambda(A_{rs})$ is the coefficient of $t_1^r t_2^s$ for all large r and s in the power series expansion of

$$(*) \qquad E(t_1,t_2) := \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{N^{(i,j)}(-1)^{i+j}}{i!j!(1-t_1)^{m+1-i}(1-t_2)^{n+1-j}}.$$

Since the coefficient of $t_1^r t_2^s$ in $E(t_1,t_2)$ is given by a polynomial for all r and s,

$$(**)\qquad E(t_1,t_2) = \sum_{(r,s)\in\mathbf{N}^2} P(r,s;A)t_1^r t_2^s.$$

Here we are using the fact that two polynomials in $\mathbf{Z}[x,y]$ coinciding at (r,s) for all large r and s must be equal. Now expand the rational functions in (*) to get

$$E(t_1,t_2) = \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{N^{(i,j)}(-1)^{i+j}}{i!j!}\left[\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{m-i+r}{r}\binom{n-j+s}{s}t_1^r t_2^s\right].$$

Comparing this with (**) yields the desired formula.

We conclude this section with an example showing that Theorem 2.4 in Amao's paper [A] is not quite correct. According to this, if S is an Artinian ring and we take the polynomial ring $R = S[X_1,\dots,X_m;Y_1,\dots,Y_n]$ over S and $M = \bigoplus_{r,s\geq 0} M_{rs}$ is a finitely generated bigraded R-module, then there exists $g(t_1,t_2)\in \mathbf{Z}[t_1,t_2]$ so that

$$\sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\lambda(M_{rs})t_1^r t_2^s = \frac{g(t_1,t_2)}{(1-t_1)^{p+1}(1-t_2)^{q+1}}$$

where $p+q+2 = \dim_R(M)$. Recall the analogous theorem for the Hilbert series of a finitely generated module $M = \bigoplus_{n\geq 0} M_n$ over a Noetherian graded ring $A = \bigoplus_{n\geq 0} A_n = A_0[A_1]$. If the dimension of M is d then for some $f(t)\in\mathbf{Z}[t]$, $\sum_{n=0}^{\infty}\lambda(M_n)t^n = f(t)/(1-t)^d$. Unfortunately, this does not extend to bigraded modules as Amao claims. This is evident from the following

Example 3.2. Let k be a field and set $R = k[X,Y;Z,W]$ be the four dimensional polynomial ring bigraded by the sets $\{X,Y\}$ and $\{Z,W\}$ in the usual way. Let $f = XZ - YW$. A bigraded resolution of $A = R/(f)$ is given by the exact sequence

$$0 \to R(-1,-1) \xrightarrow{\cdot f} R \to A \to 0.$$

It follows readily from this that the Hilbert series for A is given by

$$Q(t_1,t_2;A) = \frac{(1-t_1t_2)}{(1-t_1)^2(1-t_2)^2}.$$

According to Amao's theorem there exist $p,q\in\mathbf{N}$ and $g(t_1,t_2)\in\mathbf{Z}[t_1,t_2]$ such that

$$\frac{(1-t_1t_2)}{(1-t_1)^2(1-t_2)^2} = \frac{g(t_1,t_2)}{(1-t_1)^{p+1}(1-t_2)^{q+1}}$$

where $p+q+2 = \dim(A) = 3$. Hence $p+q=1$. Without loss of generality we may assume that $p=0, q=1$. Hence $1-t_1t_2 = (1-t_1)g(t_1,t_2)$. But this is a contradiction since $1-t_1$ does not divide $1-t_1t_2$.

Section 4. Multiplicity and Degrees.

For the bigraded S-algebra $A = S[X;Y]/I$ as above, one may also view A as a graded algebra $A = \bigoplus_{n\geq 0} A_n$ where A_n is the direct sum of the bigraded components A_{rs} with $r+s=n$. Now let $d = \dim(A)$. Then $\lambda(A_n)$ for large n is a polynomial in n of degree $d-1$. The coefficient of $n^{d-1}/(d-1)!$, denoted by $e(A)$, is called the multiplicity (or degree) of A. In this section we demonstrate the connection between $e(A)$ and the degrees of I. As a consequence, we recover a formula in [V] for the multiplicity of the Rees algebra $B[Jt]$ localized at its homogeneous maximal ideal where B is a local ring and $J \subseteq B$ is an ideal of positive height.

Theorem 4.1. *If $(X)A$ and $(Y)A$ have positive height, then*

$$e(A) = \sum_{i+j=d-2} e_{ij}.$$

Proof. As in the proof of Theorem 2.1, $\lambda(A_n) = \lambda(B_n)+\lambda(C_n)+\lambda(D_n)$. Since $\dim(A) = d$ and height$((X)A)$ and height$((Y)A)$ are positive, $\lambda(B_n)$ and $\lambda(D_n)$ are polynomials of degree at most $d-2$ for large n. Let $P(n)$ denote the Hilbert polynomial corresponding to the Hilbert function $\lambda(A_n)$. Since height$((X)A)$ and height$((Y)A)$ are positive, rdim$(A) = \dim(A) = d$. Hence the total degree of $P(r,s;A)$ is $d-2$. Continuing with the notation in the proof of Theorem 2.1, for $n \geq q$,

$$P(n) = \lambda(C_n) = \sum_{i=0}^{n-2q} P(n-q-i, q+i; A) = \sum_{i=q}^{n-q} P(n-i, i; A).$$

Write the Hilbert polynomial $P(r,s;A)$ as $\sum_{p=0}^{d-2} \frac{a_p r^p s^{d-2-p}}{p!(d-2-p)!}$ + terms of total degree $\leq d-3$. Then

$$P(n) = \sum_{i=q}^{n-q}\sum_{p=0}^{d-2} \frac{a_p (n-i)^p i^{d-2-p}}{p!(d-2-p)!} + \cdots$$

$$= \sum_{i=q}^{n-q}\sum_{p=0}^{d-2} \frac{a_p i^{d-2-p}}{p!(d-2-p)!}\left[\sum_{k=0}^{p}\binom{p}{k} n^{p-k} i^k (-1)^k\right] + \cdots$$

$$= \sum_{p=0}^{d-2}\sum_{k=0}^{p} \frac{a_p}{(d-2)!}\binom{d-2}{p}\binom{p}{k}(-1)^k n^{p-k} \sum_{i=q}^{n-q} i^{d-2-p+k} + \cdots$$

$$= \sum_{p=0}^{d-2}\sum_{k=0}^{p} \frac{a_p}{(d-2)!}\binom{d-2}{p}\binom{p}{k}(-1)^k n^{p-k}\left[\frac{n^{d-1-p+k}}{(d-1-p+k)}+\cdots\right]+\cdots$$

(by Lemma 2.8 of [V])

$$= \frac{n^{d-1}}{(d-1)!}\sum_{p=0}^{d-2} a_p \left[\sum_{k=0}^{p}\binom{d-2}{p}\binom{p}{k}\frac{(-1)^k(d-1)}{(d-1-p+k)}\right]+\cdots$$

$$= \frac{n^{d-1}}{(d-1)!}\left[\sum_{p=0}^{d-2} a_p\right]+\cdots$$

(by Lemma 2.7 of [V]). Since $a_p = e_{p,d-2-p}$, the result follows.

Example 4.2. If height($(X)A$) or height($(Y)A$) fail to be positive, the multiplicity of A may not equal the sum of the degrees of I. Let k be a field, $R = k[X_0, X_1; Y_0, Y_1, Y_2]$ and $I = (X_0Y_0, X_1Y_1)$. Then $\dim(A) = 3 = \operatorname{rdim}(A)$. Hence $\deg(P(r,s;A)) = 1$. To calculate the degrees of I, consider the bigraded Koszul complex

$$0 \to R(-2,-2) \to R^2(-1,-1) \to R \to A \to 0.$$

It follows that

$$Q(t_1,t_2;A) = Q(t_1,t_2;R) - 2t_1t_2Q(t_1,t_2;R) + t_1^2t_2^2Q(t_1,t_2;R) = \frac{(1-t_1t_2)^2}{(1-t_1)^2(1-t_2)^3}.$$

By Theorem 3.1,

$$e_{10} = \frac{(-1)^{1-2-1}}{(1-1)!(2-0)!}\left.\frac{\partial^2(1-t_1t_2)^2}{\partial t_2^2}\right|_{t_1=t_2=1} = 1$$

$$e_{01} = \frac{(-1)^{1+2-1}}{(1-0)!(2-1)!}\left.\frac{\partial^2(1-t_1t_1)^2}{\partial t_1\partial t_2}\right|_{t_1=t_2=1} = 2.$$

Similarly, one may calculate the Hilbert series of A as a graded algebra and obtain $Q(t;A) = (1+t)^2/(1-t^3)$. By Corollary 11.2 of [A-M], $e(A) = 4$. Thus $e(A) > e_{10} + e_{01}$.

In the above example, one can see that the Hilbert series of the graded algebra A can be obtained from the Hilbert series of the bigraded algebra A by putting $t_1 = t_2 = t$. This is true in general. We state this as a proposition and leave the proof to the reader.

Proposition 4.3. *Let $A = \bigoplus_{r,s\geq 0} A_{rs}$ be a bigraded algebra over an Artinian local ring A_{00}. Put $Q(t_1,t_2) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \lambda(A_{rs})t_1^r t_2^s$. For the corresponding graded algebra $A = \bigoplus_{n\geq 0} A_n$, put $P(t) = \sum_{n=0}^{\infty} \lambda(A_n)t^n$. Then $P(t) = Q(t,t)$.*

We conclude this section by giving an application of Theorem 4.1 to Rees algebras, thereby generalizing Theorem 3.1 of [V]. We now change our notation and let (R, m) be a local ring of dimension $d > 0$ and $J \subseteq R$ an ideal of positive height. Then the Rees algebra $R[Jt]$, t an indeterminate, is the graded R-algebra

$$T = R[Jt] = R \oplus Jt \oplus J^2t^2 \oplus \cdots .$$

T has a unique homogeneous maximal ideal, namely $M = (m, Jt)$. It is well-known that $\dim(T_M) = d+1$. Let I be an m-primary ideal of R. Then $(I, Jt)T$ is M-primary. We wish to obtain a formula for $e((I,Jt)T_M)$ in terms of the degrees of a certain bihomogeneous ideal. Consider the bigraded algebra

$$A = \bigoplus_{r,s\geq 0} I^rJ^s/I^{r+1}J^s$$

over the Artinian local ring $R/I = A_{00}$. To determine $e((I,Jt)T_M)$, we will need to know the total degree of the Hilbert polynomial $P(r,s;A)$ corresponding to the Hilbert function $\lambda(I^rJ^s/I^{r+1}J^s)$. Bhattacharya finds the degree of $P(r,s;A)$ in Theorem 7 of [B] to be $\dim(A) - 2$. Unfortunately the incorrect result [B, Theorem 6] is used in doing so. That the degree of $P(r,s;A)$ is $d-1$ is proved in Theorem 2.7 in [K-V] which also proves that

$$P(r,s;A) = \sum_{j=0}^{a-1} \binom{r}{d-1-j}\binom{s}{j} e_j(I|J) + \text{ terms of lower degree}$$

where $a = a(J) =$ analytic spread of $J := \dim(\bigoplus_{n\geq 0} J^n/mJ^n)$, $e_0(I|J) = e(I)$ and $e_j(I|J)$, are positive integers. (The reader should note the first and second authors' own error. While the proof of Theorem 2.7 in [K-V] correctly shows that $e_j(I|J) = 0$, $j \geq a$, $e_j(I|J) \neq 0$, $j < a$, the inequality $e_j(I|J) \geq e(J+I)$ in part (ii) of Theorem 2.7 is valid only for $0 \leq j \leq \text{height}(J) - 1$, not $0 \leq j \leq a-1$. Consequently, one must replace $a(J)$ by height(J) in [K-V] Corollaries 2.8 and 3.7.)

Theorem 4.4. $e((I,Jt)T_M) = e_0(I|J) + e_1(I|J) + \cdots e_{a-1}(I|J)$. *In particular,* $e(T_M) = e(R) + e_1(M|J) + \cdots + e_{a-1}(m|J)$.

Proof. To calculate $e((I, Jt)T_M)$, set $K = (I, Jt)$ and consider the powers of K. For any $n \geq 0$,

$$K^n = I^n \oplus I^{n-1}Jt \oplus \cdots \oplus J^n t^n \oplus J^{n+1}t^{n+1} \oplus \cdots,$$

so

$$\lambda(K^n/K^{n+1}) = \sum_{i=0}^{n} \lambda(I^{n-i}J^i/I^{n+1-i}J^i) = \sum_{r+s=n} \lambda(I^r J^s/I^{r+1}J^s).$$

Write $A_n = \bigoplus_{r+s=n} I^r J^s/I^{r+1}J^s$. Then $A = \bigoplus_{n\geq 0} A_n$ and $e(KT_M) = e(A)$. In view of Theorem 4.1, it only remains to show that A_{01} and A_{10} generate ideals of positive height. We do this by viewing A as $\mathcal{R}/I\mathcal{R}$, where $\mathcal{R} = R[It, Ju]$ (t, u indeterminates). Let $P \subseteq \mathcal{R}$ be a prime minimal over $I\mathcal{R}$. We must show $It \not\subseteq P$ and $Ju \not\subseteq P$. Since $\mathcal{R}$ is the Rees algebra of $R[Ju]$ with respect to $IR[Ju]$, it is well-known that It cannot be contained in P. Since $d > 0$ and $I\mathcal{R}_P$ is principal, height$(P) = 1$. Now, thinking of $\mathcal{R}$ as the Rees algebra of $R[It]$ with respect to $JR[It]$, it follows that $Ju \not\subseteq P$, as well (since $P \cap R[It]$ has height greater than zero). This finishes the proof.

Section 5. Examples.

Example 5.1. Let $R = k[X_0, \ldots, X_m; Y_0, \ldots, Y_n]$ be a polynomial ring in $m + n + 2$ variables over the field k, $m \geq 2$, $n \geq 2$. Let M be a $t \times (t+1)$ matrix whose entries are forms of degree (d, e), $d \geq 1$, $e \geq 1$. Let I be the ideal generated by the $t \times t$ minors of M Hence each minor is a bihomogeneous polynomial of degree (td, te). Suppose height$(I) = 2$. Then $A = R/I$ has the following bigraded resolution

$$0 \to R^t(-td-d, -te-e) \xrightarrow{M} R^{t+1}(-td, -te) \to R \to A \to 0.$$

Hence $Q(t_1, t_2; A) = (1 - (t+1)t_1^{td}t_2^{te} + tt_1^{td+d}t_2^{te+e})/(1-t_1)^{m+1}(1-t_2)^{n+1}$. Since I is an unmixed height two ideal, and $(X)A$ and $(Y)A$ have positive height, rdim$(A) = \dim(A) = m + n \geq 4$. Hence $\deg(P(r, s; A)) = m + n - 2$. By Theorem 3.1

$$e_{m,m-2} = \binom{t+1}{2}e^2, e_{m-1,n-1} = 2\binom{t+1}{2}de, e_{m-2,n} = \binom{t+1}{2}d^2.$$

Hence $e(A) = \binom{t+1}{2}e^2 + 2\binom{t+1}{2}de + \binom{t+1}{2}d^2 = \binom{t+1}{2}(d+e)^2$. This is in agreement with the formula in Example 1.5 of [H-M].

Example 5.2. Let $a_0, a_1, \ldots, a_d$ be a sequence of non-negative integers satisfying $a_0 + \cdots + a_d \geq 1$. We construct a bihomogeneous ideal in the ring $R = k[X_0, \ldots, X_d; Y_0, \ldots, Y_d]$

over a field k so that the degrees of I are $a_0, \ldots, a_d$. In other words, the Hilbert polynomial $P(r,s;A)$ of the bigraded algebra $A = R/I$ has the form

$$P(r,s;A) = \sum_{i=0}^{d} a_i \binom{r}{d-i}\binom{s}{i} + f(r,s)$$

where $f(r,s) \in \mathbb{Q}[r,s]$ is a polynomial of total degree at most $d-1$. Consider the ideals

$$q_0 = (Y_0^{a_0}, Y_1, \ldots, Y_{d-1}), q_i = (X_0^{a_i}, X_1, \ldots, X_{i-1}, Y_0, \ldots, Y_{d-1-i})$$

for $i = 1, \ldots, d$. Set $I = q_0 \cap q_1 \cap \cdots \cap q_d$ and put $J_i = (q_0 \cap \cdots \cap q_i) + q_{i+1}$, for $i = 0, \ldots, d-1$. By considering exact sequences of the form

$$0 \to R/q_1 \cap \cdots \cap q_{i+1} \to (R/q_1 \cap \cdots \cap q_i) \oplus R/q_{i+1} \to R/J_i \to 0$$

it is easily seen that

$$P(r,s;A) = \sum_{i=0}^{d} P(r,s;R/q_i) - \sum_{i=0}^{d-1} P(r,s;R/J_i).$$

Any prime containing J_i contains $q_j + q_{i+1}$ for some $j \leq i$. Hence $\dim(R/J_i) \leq d+1$. Hence $\deg(P(r,s;R/J_i) \leq d-1$. Thus to calculate the degrees of I it is enough to calculate the degrees of q_i for $i = 0, 1, \ldots, d$. If $a_i = 0$, then $q_i = R$ and $P(r,s;R/q_i) = 0$. To calculate $P(r,s;R/q_0)$, let $a_0 \geq 1$. Then $R/q_0 \cong k[X_0, \ldots, X_d; Y_o, Y_d]/(Y_0^{a_0})$. Putting $S = k[X_0, \ldots, X_d; Y_0, Y_d]$, it follows from the exact sequence

$$0 \to S(0, -a_0) \xrightarrow{\cdot Y_0^{a_0}} S \to R/q_0 \to 0$$

that

$$P(r,s;R/q_0) = \binom{r+d}{d}\binom{s+1}{1} - \binom{r+d}{d}\binom{s+1-a_0}{1} = a_0\binom{r}{d} + f_0(r),$$

where $f_0(r) \in \mathbb{Q}[r]$ has degree less than d. By a similar argument we get $P(r,s;R/q_d) = a_d\binom{s}{d} + f_d(s)$, where $f_d(s) \in \mathbb{Q}[s]$ has degree less than d. For $i = 2, \ldots, d-1$ we get

$$P(r,s;R/q_i) = a_i\binom{r}{d-i}\binom{s}{i} + f_i(r,s)$$

where $f_i(r,s) \in \mathbf{Q}[r,s]$ has total degree less than d. Hence

$$P(r,s;R/I) = \sum_{i=0}^{d} a_i \binom{r}{d-i}\binom{s}{i} + f(r,s)$$

where $f(r,s)$ denotes the terms of degree less than d.

Let I be a bihomogeneous ideal of the polynomial ring $R = S[X,Y]$, where as before, S is an Artinian local ring, $X = \{X_0,\ldots,X_m\}$, $Y = \{Y_0,\ldots,Y_n\}$ and set $A = R/I$. Suppose $\operatorname{rdim}(A) = d+2$. Write

$$P(r,s;A) = \sum_{i=0}^{d} a_i \binom{r}{d-i}\binom{s}{i} + f(r,s)$$

where $f(r,s)$ represents the terms of total degree less than d. We say that the sequence of degrees $a_0, a_1, \ldots, a_d$ is *rigid* if there exist i,j with $0 \le i \le j \le d$ such that $a_k = 0$ for $k \le i$ and $k \ge j$ and the other degrees are non-zero. For instance, in the example above, we may take $d = 2$, $a_0 = 1$, $a_1 = 0$, $a_2 = 1$ to obtain an ideal $I \subseteq k[X_0,X_1,X_2;Y_0,Y_1,Y_2]$ whose sequence of degrees is $1,0,1$ (and is therefore not rigid).

Question 5.3. If S above is a field and $A = R/I$ is Cohen-Macaulay or a domain, then is it true that the sequence of the degrees of I is rigid ? Neither of these conditions is necessary for the rigidity of the degree sequence as easy examples show. Our last example illustrates this, as well as most of the main results in this paper.

Example 5.4. C. Huneke gave the following example of a height two Cohen-Macaulay bihomogeneous ideal I in $R = k[X_0,X_1,X_2;Y_0,Y_1,Y_2]$ so that I is neither prime nor a complete intersection and the sequence of degrees of I is rigid. Consider the matrix

$$M = \begin{pmatrix} X_0 & Y_0^2 & Y_1 \\ X_1 & Y_1^2 & Y_0 \end{pmatrix}$$

and let $I = (\delta_1,\delta_2,\delta_3)$ denote the ideal of (signed) 2×2 minors of M (obtained by deleting the first, second and third rows, resp.). By the Hilbert-Burch theorem $R/I = A$ is Cohen-Macaulay. Since $Y_0^3 - Y_1^3 = (Y_0 - Y_1)(Y_0^2 + Y_1^2 + Y_0Y_1) \in I$, but neither factor belongs to I, I is not prime. Consider the bigraded resolution of A

$$0 \to R^2(-1,-3) \xrightarrow{M^t} R(0,-3) \oplus R(-1,-1) \oplus R(-1,-2) \xrightarrow{I} R \to A \to 0.$$

It follows that the Hilbert series of A is given by $N(t_1,t_2)/(1-t_1)^3(1-t_2)^3$, where $N(t_1,t_2) = 1 - t_2^3 - t_1t_2 - t_1t_2^2 + 2t_1t_2^3$. By Theorem 3.1,

$$e_{20} = \frac{1}{2!}\frac{\partial^2 N(t_1,t_2)}{\partial t_2^2}\bigg|_{t_1=t_2=1} = 2$$

$$e_{11} = \left.\frac{\partial^2 N(t_1,t_2)}{\partial t_2 \partial t_1}\right|_{t_1=t_2=1} = 3$$

$$e_{02} = \frac{1}{2!}\left.\frac{\partial^2 N(t_1,t_2)}{\partial t_1^2}\right|_{t_1=t_2=1} = 0.$$

Hence the sequence of the degrees of I is $2,3,0$, which is rigid. Additionally, since the Hilbert series of A as a graded algebra is obtained by putting $t_1 = t_2 = t$ in $Q(t_1,t_2;A)$, we get $Q(t;A) = 1 + 2t + 2t^2/(1-t)^4$. Thus $e(A)$, which is the value of $1 + 2t + 2t^2$ at $t = 1$, is $5 = e_{20} + e_{11} + e_{02}$. This is what Theorem 4.1 predicts.

Added in proof : Since the first author delivered a talk at Trieste on these matters, the comprehensive account of multigraded Hilbert functions [K-R] has come out. The interested reader is encouraged to consult this paper.

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