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Note on Ideal-Transforms, Rees Rings, and Krull Rings

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Let I be an ideal in a Noetherian ring R and let $T(I)$ be the ideal-transform of R with respect to I. Several necessary and sufficient conditions are given for $T(I)$ to be Noetherian for a height one ideal I in an important class of altitude two local domains, and some specific examples are given to show that the integral closure $T(I)'$ and the complete integral closure $T(I)''$ of $T(I)$ may differ, even when R is an altitude two Cohen–Macaulay local domain whose integral closure is a regular domain and a finite R-module. It is then shown that $T(I)''$ is always a Krull ring, and if the integral closure of R is a finite R-module, then $T(I)''$ is contained in a finite $T(I)$-module. Finally, these last two results are applied to certain symbolic Rees rings. © 1987 Academic Press, Inc

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1. Introduction

Ideal-transforms were introduced by Nagata in [12] and they proved to be very useful in his series of papers on the Fourteenth Problem of Hilbert. Also, they have been used in the study of over-rings of a given ring, the catenary chain conjectures, and asymptotic prime divisors, so they are an important and interesting area of commutative algebra. Rees rings have been useful in an auxiliary role in many research problems in commutative algebra, and quite a few papers concerning specific properties of Rees rings have recently appeared. Finally, Krull domains constitute an important area of commutative algebra and algebraic geometry.

The present paper considers two results where these three areas intersect; namely, we consider when the complete integral closure of certain ideal-transforms of certain Rees rings are Krull rings and when they are contained in a finitely generated module. To be somewhat more specific, one of the main results in [8] showed that if I is a regular ideal in a Noetherian ring such that the ideal-transform $T(I)$ is integrally closed, then $T(I)$ is a Krull ring. One of the main results in [9] showed that if Q is a P-primary ideal in a Noetherian domain R such that all the ideals $Q^n R_P$ are integrally closed, then the symbolic Rees ring $\mathbf{T} = R[u, tQ, t^2 Q^{(2)}, ...]$ (t is an indeterminate and $u = 1/t$) is a Krull domain. (It follows from [8, (4.2)] that $\mathbf{T}$ is the ideal-transform of the Rees ring $R[u, tQ]$ with respect to some ideal containing u.)

Two of the main results in this paper are related to these results. The first, (4.1), shows that the complete integral closure $T(I)''$ of $T(I)$ is always a Krull ring and that the complete integral closure $\mathbf{T}''$ of $\mathbf{T}$ is always a Krull domain. The second, (4.6), shows that if the integral closure of R is a finite R-module, then $T(I)''$ is contained in a finite $T(I)$-module, and this continues to hold for the ideal-transforms of Rees rings if R is an analytically unramified local ring.

Of course, it is of interest in the first result mentioned in the preceding paragraph whether the integral closure of $T(I)$ (resp., of $\mathbf{T}$) is a Krull ring (resp., a Krull domain), and for the second result whether the integral closure is itself a finite $T(I)$-module. So before proving (4.1) and (4.6) we first give in (3.2) several necessary and sufficient conditions for $T(I)'$ to be a Krull domain for the case when I is a height one ideal in an important class of altitude two local domains. Using this theorem, some specific examples are given to show that the integral closure of $T(I)$ need not be a Krull ring, even when I is a height one prime ideal in a very nice local domain, and a final example shows that this can even happen when I is a height one prime ideal containing u in a Rees ring, so $\mathbf{T}$ need not be a Krull domain. It then follows that neither the integral closure nor the complete integral closure of $T(I)$ need be a finite $T(I)$-module.

In closing this introduction it should be noted that a number of the results in this paper are closely related to those of Krull in [10], which is an excellent reference for this area.

2. Definitions and Known Results

This section contains the definitions needed for the remainder of the paper together with a few known results that will be needed in what follows.

(2.1) Definition. Let I be an ideal in a ring R. Then:

(2.1.1) If Q is the total quotient ring of R, then the *ideal-transform* $T(I)$ *of* R *with respect to* I (or the *I-transform* $T(I)$ *of* R) is defined by $T(I) = \{r/s \in Q;\ r, s \in R,\ (0){:}\ sR = (0)$, and $(r/s) I^n \subseteq R$ for some $n \geq 1\}$.

(2.1.2) I is said to be *regular* if it contains some regular element of R.

(2.1.3) A *filtration* $f = \{I_n\}_{n \geq 0}$ on R is a descending sequence of ideals I_n such that $I_0 = R$ and $I_i I_j \subseteq I_{i+j}$ for all $i, j \geq 0$.

(2.1.4) If $f = \{I_n\}_{n \geq 0}$ is a filtration on R, then the *Rees ring of* R *with respect to* f is the graded subring $\mathbf{R}(R, f) = R[u, tI_1, t^2 I_2, ...]$ of $R[u, t]$, where t is an indeterminate and $u = 1/t$. If f is the set of powers I^n of I, then we will write $\mathbf{R}(R, I)$ in place of $\mathbf{R}(R, f)$.

(2.1.5) A *grade one prime ideal* is a prime ideal which is a prime divisor of a regular principal ideal in R.

(2.1.6) A *Krull ring* is a ring such that: (a) $K = \bigcap \{K_{(p)};\ p$ is a grade one prime ideal in $K\}$, where (p) is the set of regular elements in $K - p$; (b) $K_p/\mathrm{Rad}(K_p)$ is a discrete valuation ring for all grade one prime ideals p in K; and, (c) each regular element in K is contained in only finitely many grade one prime ideals in K.

(2.1.7) R' will be used to denote the *integral closure* of R, and R'' will denote the *complete integral closure* of R.

(2.2) *Remark.* (2.2.1) It is well known that $T(I)$ is an overring of R (that is, a ring between R and its total quotient ring) and that if R is an integral domain, then $T(I) = \bigcap \{R_p;\ p \in \mathrm{Spec}(R), \mathrm{grade}(p) = 1$, and $I \nsubseteq p\}$ and, if $I = (b_1, ..., b_n) R$, then $T(I) = \bigcap \{R_{b_i};\ i = 1, ..., n\}$.

(2.2.2) It is clear that a Krull domain is a Krull ring, and it is shown in [15] that the integral closure of a Noetherian ring is a Krull ring.

(2.2.3) It was shown in [8, (2.6)] that if R is either a Noetherian ring or a Krull ring, then the following is true: if b is a regular element in an ideal I in R and if c is a regular element in I such that for each grade one prime ideal p in R that contains bR, $c \in p$ if and only if $I \subseteq p$, then $T(I) = T((b, c)R)$. It follows from this that if I is a regular ideal in a Noetherian ring R, then there exist regular nonunits $b, c \in I$ such that $T(I) = T((b, c)R)$ and $T(IR') = T((b, c)R')$.

(2.2.4) It was shown in [8, (3.3)] that if R is either a Noetherian ring or the integral closure of a Noetherian ring and I is a regular ideal in R such that $T(I)$ is integrally closed, then $T(I)$ is a Krull ring.

(2.2.5) It was shown in [9, Theorem B] that if R is an integrally closed Noetherian domain and Q is a P-primary ideal such that $Q^n R_P$ is integrally closed for all n, then $\mathbf{T} = R[u, tQ, t^2Q^{(2)}, ...]$ is a Krull domain.

(2.2.6) Let I be an ideal in a Noetherian ring R, let $\mathbf{R} = \mathbf{R}(R, I)$, and let S be a saturated multiplicatively closed set of regular elements in $\mathbf{R}$ such that $u \notin S$. Let $f = \{I_n\}_{n \geqslant 0}$, where $I_n = u^n \mathbf{R}_S \cap R$, and let $\mathbf{T} = \mathbf{R}(R, f)$. Then it is shown in [8, (4.2)] that $\mathbf{T} = T((u, c)\mathbf{R})$, where c is a regular element in $\mathbf{R}$ that is in a grade one prime p containing u if and only if $p \cap S \neq \emptyset$.

3. When Is $T(I)$ Finitely Generated over R?

The main result in this section, (3.2), gives several necessary and sufficient conditions for the ideal-transform $T(I)$ to be finitely generated over R, when I is a height one ideal in an important class of altitude two local domains. Our results in Section 4 are concerned with $T(I)''$, so we note here that in the case considered in (3.2), $T(I)$ is finitely generated over R if and only if $T(I)' = T(I)''$.

Lemma (3.1) will be used several times in the proof of (3.2). In the proof of (3.1) we use the following fact: if $R \subseteq S$ are integral domains such that R is Noetherian, then the altitude inequality holds between S and R; that is, if P is a prime ideal in S, and if $p = P \cap R$, then $\text{height}(P) + \text{trd}((S/P)/(R/p)) \leqslant \text{height}(p) + \text{trd}(S/R)$. This is proved in [3, Theorem 1], although it is stated and proved in most texts on Commutative Algebra only in the case where S is finitely generated over R. We are indebted to the referee for reminding us of Ref. [3]; this helped shorten our original proof of (3.1).)

(3.1) Lemma. *Let (R, M) be an integrally closed and analytically irreducible local domain of altitude two, let I be a proper nonzero ideal in R, and let $T = T(I)$. Then either* $\text{height}(I) = 2$ *and* $T = R$ *or* $\text{height}(I) = 1 = \text{altitude}(T)$ *and T is finitely generated over R.*

Proof. $T = \bigcap \{R_p;\, p \in \mathrm{Spec}(R)$ and $I \not\subset p\}$, so if height$(I) = 2$, then $T = R$, since $R = \bigcap \{R_p;\, p \in \mathrm{Spec}(R)$ and height$(p) = 1\}$. Therefore it may be assumed that height$(I) = 1$.

Suppose that altitude$(T) = 2$ and let M^* be a height two maximal ideal in T, so necessarily $M^* \cap R = M$. Now T is a Krull domain, by [13, Lemma 2.4], so $L = T_{M^*}$ is an altitude two Krull domain between R and its quotient field F, so L is Noetherian, by [6, Theorem 9]. Also, height$(IT) \neq 1$, by [13, Corollary, p. 113], so it follows that ML is M^*L-primary. Further, it follows immediately from the altitude inequality (see the comment preceding this lemma) that T/M^* is algebraic over R/M, and it then follows from [3, Theorem 3] that $L/M^*L = T/M^*$ is finitely generated over R/M. Therefore L/ML is finitely generated over R/M, since ML is M^*L-primary. Thus since R is analytically irreducible it follows from [14, (37.4)] that $L = R$, so $T = R$, and this contradicts the fact that height$(I) = 1$ and height$(IT) > 1$. Therefore altitude$(T) = 1$. Thus, since no height one prime ideal in T contains IT it follows that $IT = T$. Let $1 = \sum_1^n i_j t_j$, where $i_j \in I$ and $t_j \in T$ $(j = 1, \ldots, n)$, and let $A = R[t_1, \ldots, t_n]$. Then $IA = A$, so $T(IA) = T(A) = A$, and $T(IA) = T$, by [13, Lemma 2.5], so $A = T$, hence T is finitely generated over R. Q.E.D.

The next theorem, (3.2), is the main result in this section; it gives several necessary and sufficient conditions for $T(I)$ to be finitely generated over R for height one ideals I in certain altitude two local domains.

(3.2) THEOREM. *Let (R, M) be a Cohen–Macaulay local domain of altitude two such that R' is a finite R-module and $R'_{M'}$ is analytically irreducible for all maximal ideals M' in R'. Let I be a height one ideal in R, let $T = T(I)$, and let $T^* = T(IR')$. Then the following statements are equivalent:*

(3.2.1) *T is finitely generated over R.*

(3.2.2) *T is Noetherian.*

(3.2.3) $T' = T''$.

(3.2.4) $T' = T^*$.

(3.2.5) $IT = T$.

(3.2.6) Altitude$(T) = 1$.

(3.2.7) *T is a flat R-module.*

(3.2.8) Altitude$(IR') = 1$ (*that is, each minimal prime divisor of IR' has height one*).

Proof. It is clear that (3.2.1) $\Rightarrow$ (3.2.2), and (3.2.2) $\Rightarrow$ (3.2.3), since the integral closure of a Noetherian domain is a Krull domain (and hence is completely integrally closed).

Assume that (3.2.3) holds, let c be a nonzero element in the conductor $R : R'$ of R in R', and let $t \in T^*$. Then it is readily seen that $ct^n \in T$ for all $n \geqslant 1$, so $T^* \subseteq T''$, and so $T^* = T''$, since T^* is a Krull domain, by [13, Lemma 2.4]. Therefore $(3.2.3) \Rightarrow (3.2.4)$.

Assume that (3.2.4) holds. Now R' has only finitely many maximal ideals, so it follows from (3.1) that T^* is finitely generated over R', so T^* is finitely generated over R, since R' is. Therefore it follows from the Artin–Tate Theorem (e.g., [1, Proposition 7.8]) that T is finitely generated over R, so $(3.2.4) \Rightarrow (3.2.1)$.

Assume that (3.2.1) holds and suppose that (3.2.5) does not hold. Then $\text{height}(IT) \geqslant 2$, by either [12, Corollary, p. 62] or [16, Lemma 5.6] (since $T = \bigcap \{R_p; p \in \text{Spec}(R) \text{ and } I \nsubseteq p\}$). Let N be a maximal ideal in T that contains IT. Now T is an over-ring of an altitude two Noetherian domain, so $\text{altitude}(T) \leqslant 2$, and so $\text{height}(N) = 2$. Also, $M = N \cap R$, so N is isolated over M (i.e., N is both a maximal and a minimal prime divisor of MT). Let $A = R' \cap T$, so A is integrally closed in T. Also, $T = T(IA)$, by [13, Lemma 2.5], and N is isolated over $N \cap A$ (since $R \subseteq A \subseteq R'$ and N is isolated over M). Therefore it follows from (3.2.1) and the Peskine–Evans version of Zariski's Main Theorem [4] that $T_N = A_{N \cap A}$, so $\text{height}(IA_{N \cap A}) = 2$. We now show that this cannot happen. If $N \cap A$ is the only maximal ideal in A, then it follows that $T = A$, $\text{height}(IT) = 2$, and $\text{height}(IA) = 1$ (since $R \subseteq A \subseteq R'$ and $\text{height}(I) = 1$), and this clearly cannot happen. Therefore there exists a nonunit $x \in A - (N \cap A)$ such that x is in every height one prime divisor of IA. Let $p_1, ..., p_k$ be the minimal prime divisors of I and let $W = \bigcap \{R_{p_i}: i = 1, ..., k\}$. Now height one prime ideals in R' contract in R to height one prime ideals, since R is Cohen–Macaulay, so it follows that $W' = \bigcap \{R'_{p'}; p' \text{ is a height one prime divisor of } p_i R' \text{ for some } i = 1, ..., k\}$, so x is in the Jacobson radical of W'. Also, $W[x]$ is a finite W-module and x is in its Jacobson radical, so $\text{altitude}(W) = 1$ implies that x^n is in the conductor $W : W[x]$ of W in $W[x]$. Therefore $x^n \in A \cap W \subseteq T \cap W = R$ (since $R = \bigcap \{R_p; p \text{ is a height one prime ideal in } R\}$). But this implies that $x^n \in M = N \cap R$, so $x \in N$ in contradiction to the choice of x. Therefore $IT = T$, so $(3.2.1) \Rightarrow (3.2.5)$.

$(3.2.5) \Rightarrow (3.2.6)$, since if $IT = T$, then $MT = T$, so it follows that $\text{altitude}(T) = 1$. And $(3.2.6) \Rightarrow (3.2.5)$, since either $IT = T$ or $\text{height}(IT) > 1$, by either [12, Corollary, p. 61] or [16, Lemma 5.6].

$(3.2.5) \Leftrightarrow (3.2.7)$, by [17, Theorem 1], and $(3.2.7) \Rightarrow (3.2.2)$, by [17, Corollary, p. 796].

Assume that (3.2.5) holds and suppose that (3.2.8) does not hold. Therefore there exists a minimal prime divisor P of IR', such that $\text{height}(P) = 2$. Then IR'_P is IR'_P-primary, so $T(IR'_P) = R'_P$, by (3.1). Let $S = R' - P$. Then $T^*_S = T(IR'_P)$, by [13, Lemma 2.6], so $T^*_S = R'_P$. But this implies that $IT^* \neq T^*$, and this contradicts (3.2.5), so $(3.2.5) \Rightarrow (3.2.8)$.

Finally, assume that (3.2.8) holds. Then to show that (3.2.5) holds assume first that $I \subseteq \mathrm{Rad}(C)$, where C is the conductor of R in R'. Then since $T = \bigcap \{R_p;\ p \in \mathrm{Spec}(R) \text{ and } I \not\subset p\}$ it follows that $T = T^*$. But since altitude$(IR') = 1$ it follows from (3.1) that altitude$(T^*) = 1$. Therefore, since IT is not contained in any height one prime ideal in T it follows that (3.2.8) $\Rightarrow$ (3.2.5) in this case.

Therefore assume that $I \not\subset \mathrm{Rad}(C)$ and let $q_1, \ldots, q_n$ be the height one prime ideals in R that contain C and do not contain I. (Note that either height$(C) = 1$ or $R = R'$, since R is Cohen–Macaulay. And if $R = R'$, then (3.2.8) $\Rightarrow$ (3.2.5) by (3.1).) Let $x \in I - \bigcup \{q_j;\ j = 1, \ldots, n\}$ and let $p_1, \ldots, p_m$ be the (height one) prime divisors of xR that do not contain I. Then $T = R_S \cap R[1/x]$, where $S = R - \bigcup \{p_i;\ i = 1, \ldots, m\}$. Therefore $xT = x(R_S \cap R_x) = xR_S \cap R_x = xR_S \cap T$, so the primary decomposition of xR_S shows that the prime divisors of xT are the ideals $p_i R_S \cap T$, $i = 1, \ldots, m$. Now altitude$(T^*) = 1$, by hypothesis and (3.1), and $T^* \subseteq R_S$, since $C \not\subset \bigcup \{p_i;\ i = 1, \ldots, m\}$ (by the choice of x and the p_i), so each $p_i R_S \cap T^*$ is a height one maximal ideal. Now by the choice of x and the p_i there exists $c \in C - p_i$. Fix $i = 1, \ldots, m$, let $p^* = p_i R_S \cap T^*$, and let $p' = p^* \cap T$. Then $cT^* \subseteq T$, so $(c + p')(T^*/p^*) \subseteq T/p'$; so since T^*/p^* is a field it follows that T/p' is. Therefore each of the prime divisors of xT is a height one maximal ideal. But $x \in I$ and no height one prime ideal in T contains IT, so it follows that $IT = T$, so (3.2.8) $\Rightarrow$ (3.2.5). Q.E.D.

(3.3) *Remark.* (3.3.1) It should be noted that (3.2.8) is equivalent to: if M' is a maximal ideal in R', then there exists a height one prime ideal $p' \subset M'$ such that $IR' \subseteq p'$, and then height$(p' \cap R) = 1$, since R is Cohen–Macaulay. With this in mind it is interesting to note that the "going-down" property between R and R' is tied in with the existence of an ideal-transform T of R such that T is not finitely generated over R. That is, if $I = p$ is a prime ideal in R, then for each maximal ideal M' in R' there exists $p' \in \mathrm{Spec}(R')$ such that $p' \subseteq M'$ and $p' \cap R = p$ if and only if T is finitely generated over R, by (3.2.1) $\Leftrightarrow$ (3.2.8).

(3.3.2) The assumption in (3.1) and (3.2) that $R'_{M'}$ is analytically irreducible is necessary. Specifically, by using the ring in [14, (E7.1), p. 210] it is shown in [7, Example 2] that there exists an altitude two analytically unramified normal local domain (R, M) such that between R and its quotient field there exists an altitude two normal local domain (S, N) such that $R < S$ and MS is N-primary. It follows that there exists a proper subset $\mathbf{P}$ of $\mathrm{Spec}(R) - \{M\}$ such that $S = \bigcap \{R_p;\ p \in \mathbf{P}\}$, so if $p \in \mathrm{Spec}(R) - \mathbf{P}$, then $T = T(p) \subseteq S$. Therefore altitude$(T) = 2$, so (3.1) fails. Also, (3.2) fails, since $pT \neq T$, but $T = T' = T''$ (since R is a Krull domain).

(3.3.3) With (3.3.2) in mind, we do not know the answer to the following question: If (R, M) is a normal local domain of altitude two whose completion has more than one minimal prime divisor of zero, then does there exist an altitude two normal local domain (S, N) such that MS is N-primary and $R < S < F$, where F is the quotient field of R? (However, if there exists a minimal prime ideal z in R^* and a height one prime ideal p in R such that $\text{height}(pR^* + z) > 1$, then there exists such a normal local domain.)

(3.3.4) An alternate proof of (3.2.8) $\Rightarrow$ (3.2.5) can be given by using Chevalley's Theorem (e.g., [5, [6.7.1)]]) concerning when a quasi-affine scheme is an affine scheme. Specifically, it holds in general that if I is an ideal in a Noetherian domain R and $T = T(I)$, then the quasi-affine scheme $\text{Spec}(R) - V(I)$ is affine if and only if $IT = T$. Therefore assume that R' is a finite integral extension of R, let $X = \text{Spec}(R') - V(IR')$ and $Y = \text{Spec}(R) - V(I)$. and let $f: X \to Y$ be the induced finite morphism of Noetherian schemes. It $T^* = T(IR')$ and $IT^* = T^*$, then X is affine. Hence by Chevalley's Theorem Y is affine and so $IT = T$.

Our first corollary of (3.2) gives a nice characterization of when a local domain R as in (3.2) has a nonfinitely generated ideal-transform.

(3.4) COROLLARY. *With R as in* (3.2), *there exists an ideal-transform $T = T(I)$ of R such that T is not finitely generated over R if and only if R' has more than one maximal ideal.*

Proof. If I is M-primary, then $T(I) = \bigcap \{R_p;\ p$ is a height one prime ideal in $R\} = R$, since R is Cohen–Macaulay, so T is finitely generated over R. And if R' is local, then $\text{altitude}(IR') = 1$ for all height one ideals I in R, so T is finitely generated over R, by (3.2.8) $\Rightarrow$ (3.2.1).

Conversely, if R' has two maximal ideals, say P and Q, then let p' be a height one prime ideal in R' such that $R: R' \not\subseteq p'$ and $p' \not\subseteq Q$. Then if $I = p' \cap R$, $\text{height}(IR'_Q) = 2$, since p' is the only prime ideal in R' lying over $p' \cap R$, so $\text{altitude}(IR') = 2$, and so T is not finitely generated over R, by (3.2.1) $\Rightarrow$ (3.2.8). Q.E.D.

The next corollary gives a global version of (3.2).

(3.5) COROLLARY. *Let R be an altitude two Cohen–Macaulay domain such that for all maximal idelals M in R it holds that $(R_M)'$ is finitely generated over R_M and $R'_{M'}$ is analytically irreducible for all maximal ideals M' in R'. Then every ideal-transform of R is finitely generated over R if and only if $(R_M)'$ is local for all maximal ideals M in R.*

Proof. It is shown in [2, Theorem 6] that $T(I)$ is finitely generated over R if and only if $T(IR_M)$ is finitely generated over R_M for all maximal ideals M in R, so this follows immediately from (3.4). Q.E.D.

Corollary (3.6) is an important special case of (3.5).

(3.6) Corollary. *Let A be a Noetherian domain of altitude one such that A' is a finite A-module and let $R = A[Y]$. Then all ideal-transforms of R are finitely generated over R if and only if there exists a unique maximal ideal in A' lying over each maximal ideal in A.*

Proof. It is readily verified that $A[Y]$ satisfies the hypotheses on R in (3.5), so this follows immediately from (3.5). Q.E.D.

(3.7) *Remark.* It follows immediately from (3.6) that if F is a field and X, Y are indeterminates, then: (a) every ideal-transform of $F[X^2, X^3, Y]$ is Noetherian; and (b) there exists an ideal-transform of $F[X(X-1), X^2(X-1), Y]$ that is not Noetherian.

Before giving one further example, we first give a brief geometric interpretation of (3.7)(b). For this, let R_0 be the coordinate algebra of the irreducible singular cubic curve C whose equation is $X^3 + XY - Y^2 = 0$ and which has a double point at the origin. Let $R = R_0 \otimes_F F[Y]$, so R is the coordinate algebra of $C \times F^1 = S$. S can be viewed as taking the affine plane $F^2 = F^1 \times F^1$ and identifying the lines $X = 0$ and $X = 1$. Then the projection mapping $\pi: F^2 \to S$ corresponds to the mapping $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$.

Now if we let l be the line $X = Y$, then l corresponds to the prime ideal $(X - Y)R'$, and $\pi(l)$ corresponds to $(X - Y) R' \cap R$. The points of F^2 which lie over $\pi(l)$ are the points of $\pi^{-1}(\pi(l))$, which we see consist not only of l, but also of the points $(0, 1)$ and $(1, 0)$.

In (3.7)(b), $R = F[X(X-1), X^2(X-1), Y]$ and we take the points $A = (0, 0)$ and $B = (1, 0)$ in F^2 which correspond to the maximal ideals $(X, Y) R'$ and $(X - 1, Y) R'$, respectively. Since no component of $\pi^{-1}(\pi(l))$ goes through B, there is no irreducible curve of F^2 which passes through B and projects onto $\pi(l)$. That is, there is no height one prime ideal of R' which is contained in $(X - 1, Y)R'$ and lies over $(X - Y)R' \cap R$.

One can view the fact that the ideal-transform of C at $(X - Y)R' \cap R$ is not affine as follows. If it were affine, then $S - \pi(l)$ would be an affine variety. But the normalization of $S - \pi(l)$ would then have to be $F^2 = \{l; (0, 1), (1, 0)\}$, and this is not an affine variety.

This section will be closed by giving one additional specific example showing that $T(I)$ need not be Noetherian. We feel that (3.8) is needed, since some of the main results in Section 4 involve the complete integral closure of an ideal-transform of a Rees ring. The reason for specifying $u \in p$ in (3.8) is that in our applications to Rees rings this condition will always hold.

(3.8) Example. There exists a local domain (L, M) that contains an ideal I such that $\mathbf{R} = \mathbf{R}(L, I)$ contains a prime ideal p such that $u \in p$ and $T(p)' \neq T(p\mathbf{R}')$, so $T(p)'$ is not a Krull domain.

Proof. Let $R \subseteq R'$ be as in (3.7)(b), let $M = (X^2 - X, X^3 - X^2, Y)R$, and let $L = R_M$, so $L' = R'_{R-M}$. Then L is a Cohen–Macaulay local domain and L' is a finite L-module and is a regular domain with exactly two maximal ideals $P = (X, Y)L'$ and $Q = (X-1, Y)L'$. Assume that the characteristic of F is two. Let $I = (X^2 + X + Y^2 + Y)L$, $\mathbf{R} = \mathbf{R}(L, I)$, $\mathbf{S} = \mathbf{R}(L', IL')$, $\mathbf{N} = (ML[u, t] \cap \mathbf{R}, u)\mathbf{R}$, $\mathbf{P} = ((X, Y)L'[u, t] \cap \mathbf{S}, u)\mathbf{S}$, and $\mathbf{Q} = ((X+1, Y)L'[u, t] \cap \mathbf{S}, u)\mathbf{S}$. Now $X^2 + X + Y^2 + Y = (X+Y)(X+Y+1)$, since $2 = 0$ in L', so $\mathbf{S}$ is a regular domain and is a finite $\mathbf{R}$-module (so $\mathbf{S} = \mathbf{R}'$), and $\mathbf{R}$ is a Cohen–Macaulay domain. Also, $X^2 + X \in L : L'$, $Y \notin L : L'$, and $Y + 1$ is a unit in L, so $X^2 + X + Y^2 + Y \notin L : L'$, so if q is a prime ideal in L that contains both $L : L'$ and $X^2 + X + Y^2 + Y$, then $(X^2 + X, Y)L \subseteq q$, so $q = ML$. Further, $p' = (X + Y, u)\mathbf{S}$ and $(X + Y + 1, u)\mathbf{S}$ are the prime divisors of $u\mathbf{S}$, $p' \subseteq \mathbf{P}$, $p' \not\subseteq \mathbf{Q}$, p' is the only prime ideal in $\mathbf{S}$ that lies over $p = p' \cap \mathbf{R}$, and p does not contain the conductor of $\mathbf{R}$ in $\mathbf{R}'$ (since $X^2 + X + Y^2 + Y \in p \cap L$ and $p \cap L \neq ML$). Let $T = T(p)$ and $T^* = T(pS)$. Then by the Cohen–Macaulay property it follows from (2.2.1) that $(*)$: $T = \bigcap \{\mathbf{R}_m; m \in \operatorname{Spec}(\mathbf{R}), \operatorname{height}(m) = 1$, and $p \not\subseteq m\}$ and $T^* = \bigcap \{\mathbf{S}_{m'}; m' \in \operatorname{Spec}(\mathbf{S}), \operatorname{height}(m') = 1$, and $p\mathbf{S} \not\subseteq m'\}$. Therefore T^* is a Krull domain, and $T^* = T(p')$, since p' is the only height one prime ideal in $\mathbf{S}$ that contains $p\mathbf{S}$. Thus since $\mathbf{S}$ is a UFD it follows from (2.2.1) that $T^* = \mathbf{S}[1/\pi]$, where $p' = \pi\mathbf{S}$, so T^* is finitely generated over $\mathbf{R}$. Therefore if T^* is integral over T, then it follows from [1, Proposition 7.8] that T is finitely generated over $\mathbf{R}$. We want to show that $T' \neq T^*$, so suppose, on the contrary, that T^* is integral over T (and so T is finitely generated over $\mathbf{R}$). Now $p' \not\subseteq \mathbf{Q}$, so $\mathbf{Q}^* = \mathbf{Q}\mathbf{S}_{\mathbf{Q}} \cap T^*$ is a maximal ideal and is the only prime ideal in T^* that lies over $\mathbf{N}$, since $\mathbf{R} \subseteq \mathbf{S} \subseteq T^*$, so it follows that $\mathbf{N}T$ is primary for $\mathbf{Q}^* \cap T$. Also, $\mathbf{R} = T \cap \mathbf{R}_p$, by $(*)$, and since $\mathbf{R}$ is Cohen–Macaulay, and $\mathbf{S} \subseteq \mathbf{R}_p$, so $\mathbf{R} = T \cap \mathbf{S}$; that is, $\mathbf{R}$ is integrally closed in T. Therefore, since T is finitely generated over $\mathbf{R}$ it follows from the Peskine–Evans version of the Zariski Main Theorem [4] that $T_{\mathbf{Q}^* \cap T} = \mathbf{R}_{\mathbf{N}}$. But this contradicts the fact that no height one prime ideal in T lies over p, hence $T' \neq T^*$. Finally, it follows from (4.1) that $T'' = T^*$, so it follows that T' is not a Krull domain. Q.E.D.

4. On the Complete Integral Closure of $T(I)$

Our original intention when starting this paper was to see if the hypothesis "$T(I)$ is integrally closed" in (2.2.4) would be omitted (that is, is $T(I)'$ always a Krull ring?), and, if not, then was it at least true that the hypotheses in (2.2.5) concerning the ideals $Q^n R_P$ can be omitted (that is, is $\mathbf{T}'$ always a Krull domain?). (It follows from [8, (4.2)] (see (2.2.6)) that $\mathbf{T}$ is an ideal-transform of $R[u, tQ]$.) It was shown in (3.7) (together with (3.2)) that the answer to the first equation is no, and (3.8) shows that the same answer applies to the second question. In this section it will be shown that affirmative answers apply if $T(I)'$ and $\mathbf{T}'$ are replaced with $T(I)''$ and $\mathbf{T}''$, respectively.

(4.1) THEOREM. *If I is a regular ideal in a Noetherian ring R, then $T(I)'' = T(IR')$ is a Krull ring.*

Proof. By (2.2.3) there exist regular elements $b, c \in I$ such that $T(I) = R_b \cap R_c$ and $T(IR') = (R')_b \cap (R')_c$. Therefore $T(IR')$ is integrally closed, so $T(IR')$ is a Krull ring, by (2.2.4), hence $T(IR')$ is completely integrally closed, by [15, Proposition 2.5]. Also, $(R')_b = (R_b)'$ and $(R')_c = (R_c)'$, and $(R_b)' \cap (R_c)' \subseteq (R_b \cap R_c)''$ (since if $x \in (R_b)' \cap (R_c)'$ and $x = r/s$ (with $r, s \in R$), then for all large n and for all $k \geqslant 1$ it holds that $s^n x^k \in R_b \cap R_c$ and s^n is regular in $R_b \cap R_c$). Therefore, since $(R_b)' \cap (R_c)' = T(IR') = T(IR')''$, it follows that $T(I)'' = (R_b \cap R_c)'' = (R')_b \cap (R')_c = T(IR')$, so $T(I)'' = T(IR')$ is a Krull ring. Q.E.D.

(4.2) COROLLARY. *Let I be an ideal in a Noetherian ring R, let $\mathbf{R} = \mathbf{R}(R, I)$, and let S be a saturated multiplicatively closed set of regular elements in $\mathbf{R}$ such that $u \notin S$. Let $I_n = u^n \mathbf{R}_S \cap R$ and let $\mathbf{T} = R[u, tI_1, t^2 I_2, ...]$. Then $\mathbf{T}''$ is a Krull ring.*

Proof. By (2.2.6), $\mathbf{T} = T((u, c)\mathbf{R})$, where c is a regular element in $\mathbf{R}$ which is in a prime divisor p of $u\mathbf{R}$ if and only if $p \cap S \neq \emptyset$. Therefore $\mathbf{T}''$ is a Krull ring by (4.1). Q.E.D.

With (3.8) in mind, (4.3) can be viewed as a generalization of (2.2.5).

(4.3) COROLLARY. *Let P be a regular prime ideal in a Noetherian ring R, let Q be a P-primary ideal, and let $\mathbf{T} = R[u, tQ, t^2 Q^{(2)}, ...]$. Then $\mathbf{T}''$ is a Krull ring.*

Proof. Let $\mathbf{R} = \mathbf{R}(R, Q)$ and let S be the set of regular elements in $R - P$. Then it is readily seen that $\mathbf{R}_S = \mathbf{R}(R_S, QR_S)$, so $Q^{(n)} = u^n \mathbf{R}_S \cap R$, so $\mathbf{T}''$ is a Krull ring, by (4.2). Q.E.D.

Before giving one more corollary of (4.1) we first note the following example which is related to (4.3).

(4.4) EXAMPLE. There exists an integrally closed complete local domain R of altitude two such that if q is a height one primary ideal in R, then the symbolic Rees ring $\mathbf{T} = R[u, tq, t^2 q^{(2)}, ...]$ is a non-Noetherian Krull domain of altitude three.

Proof. To start with we note the following three results: (a) it follows from [11, Theorem 3.1] that there exists an integrally closed complete local domain (R, M) of altitude two such that R/M is infinite and $\mathrm{Rad}(bR)$ is not prime for all nonzero nonunits b in R; (b) if R is an analytically unramified and quasi-unmixed local domain of altitude two and q is a height one primary ideal in R, then $\mathbf{T} = R[u, tq, t^2q^{(2)}, ...]$ is Noetherian if and only if $l(q^{(k)}) = 1$ for some $k \geqslant 1$, by [9, Theorem A] (where l denotes analytic spread); and (c) if R is an integrally closed local domain and q is a height one primary ideal in R, then $\mathbf{T}$ is a Krull domain if and only if q^nR_p is integrally closed for all $n \geqslant 1$, by [9, Theorem B]. Therefore let R be as in (a), let p be a height one prime ideal in R, and let q be a p-primary ideal in R. Then q^nR_p is integrally closed for all $n \geqslant 1$, since R_p is a discrete valuation ring, so $\mathbf{T}$ is a Krull domain by (c). Also, since altitude$(R) = 2$ it follows from the structure of $\mathbf{T}$ that altitude$(\mathbf{T}) = 3$. Suppose that $\mathbf{T}$ is Noetherian. Then by (b) there exist $k \geqslant 1$ such that $l(q^{(k)}) = 1$. Therefore since R/M is infinite there exists $b \in q^{(k)}$ such that $bR \subseteq q^{(k)} \subseteq (bR)_a$, where $(bR)_a$ is the integral closure of bR. But $bR = (bR)_a$, since R is integrally closed, so $q^{(k)} = bR$ is a principal ideal, and this contradicts (a). Q.E.D.

Corollary (4.5) extends (4.3). It should be noted that it holds, for example, if $P_1, ..., P_g$ are the minimal prime divisors of I and, in particular, it holds for $I = P_1^{(e_1)} \cap \cdots \cap P_g^{(e_g)}$, where the P_i are arbitrary noncomparable prime ideals in R and the e_i are arbitrary positive integers.

(4.5) Corollary. *Let I be a regular ideal in a Noetherian ring R and let $P_1, ..., P_g$ be noncomparable prime divisors of I. Let $I_n = (I^nR_{P_1} \cap R) \cap \cdots \cap (I^nR_{P_g} \cap R)$ and let $\mathbf{T} = R[u, tI_1, t^2I_2, ...]$. Then $\mathbf{T}''$ is a Krull ring.*

Proof. With S the set of regular elements in $R - \bigcup \{P_i; 1, ..., g\}$ the proof is similar to the proof of (4.3). Q.E.D.

We next consider when $T(I)''$ is contained in a finite $T(I)$-module.

(4.6) Theorem. *If R is a Noetherian ring such that R' is a finite R-module and if I is a regular ideal in R, then $T(I)'' = T(IR')$ is contained in a finite $T(I)$-module and is a Krull ring.*

Proof. By (2.2.3) there exist regular elements $b, c \in I$ such that $T(I) = R_b \cap R_c$ and $T(IR') = (R')_b \cap (R')_c$. Let x be a regular element in the conductor $R : R'$. Then $x((R_b)' \cap (R_c)') = x(R_b)' \cap x(R_c)' \subseteq R_b \cap R_c$, so it follows that $(R_b)' \cap (R_c)'$ is contained in a finite $R_b \cap R_c$-module. But $T(I)'' = (R_b)' \cap (R_c)'$, by the proof of (4.1), so the conclusion follows from (4.1), since $T(I) = R_b \cap R_c$. Q.E.D.

Concerning (4.6), it follows from (3.6) that $T(I)''$ need not be a finite $T(I)$-module, and then it is clear that $T(I)'$ is not a finite $T(I)$-module.

(4.7) Corollary. *Let R be a locally analytically unramified Noetherian ring such that R' is a finite R-module, let I be an ideal in R, and let*

$\mathbf{R} = \mathbf{R}(R, I)$. *Then for all regular ideals H in $\mathbf{R}$ it holds that $T(H)'' = T(H\mathbf{R}')$ is contained in a finite $T(H)$-module and is a Krull ring.*

Proof. It is shown in [9, Lemma 1] that the hypothesis on R implies that $\mathbf{R}'$ is a finite $\mathbf{R}$-module, so the conclusion follows immediately from (4.6). Q.E.D.

(4.8) COROLLARY. *Let R be a locally analytically unramified Noetherian ring such that R' is a finite R-module, let I be an ideal in R, and let $\mathbf{R} = \mathbf{R}(R, I)$. Let S be a multiplicatively closed set of regular elements in $\mathbf{R}$ such that $u \notin S$, let $I_n = u^n\mathbf{R}_S \cap R$, and let $\mathbf{T} = R[u, tI_1, t^2I_2, ...]$. Then $\mathbf{T}''$ is contained in a finite $\mathbf{T}$-module and is a Krull ring.*

Proof. By (2.2.6), $\mathbf{T} = T((u, c)\mathbf{R})$ for some regular element c in $\mathbf{R}$, so the conclusion follows immediately from (4.7). Q.E.D.

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