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Asymptotic prime divisors of torsion-free symmetric powers of modules

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Abstract

Let R be a Noetherian ring, $F := R^r$ and $M \subseteq F$ a submodule of rank r. Let $\overline{A^*}(M)$ denote the stable value of $\mathrm{Ass}(F_n/\overline{M_n})$, for n large, where F_n is the nth symmetric power of F_n and M_n is the image of the nth symmetric power of M in F_n. We provide a number of characterizations for a prime ideal to belong to $\overline{A^*}(M)$. We also show that $\overline{A^*}(M) \subseteq A^*(M)$, where $A^*(M)$ denotes the stable value of $\mathrm{Ass}(F_n/M_n)$.

Keywords: Prime divisor; Integral closure; Rees ring; Rees valuation

1. Introduction

Let R be a Noetherian ring, F a free R-module of rank r and $M \subseteq F$ a submodule. Write F_n for the nth symmetric power of F and M_n for the canonical image of the nth symmetric power of M in F_n. When M has a rank, e.g., if R is a domain, M_n is called the nth torsion-free symmetric power of M. In [3] it was shown that the associated primes of the modules F_n/M_n and $F_n/\overline{M_n}$ are stable for large n. Here, $\overline{M_n}$ denotes the integral closure of M_n in F_n. As is well known, there are corresponding results for ideals due to Brodmann and Ratliff, respectively. A good reference for the ideal case is McAdam's monograph [5]. In this paper we give a number of characterizations for a prime to ideal belong to the stable set of primes asso-

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ciated to $\mathrm{Ass}(F_n/\overline{M_n})$. Let $\overline{A^*}(M)$ denote this stable value. Our main result along these lines is that for a prime $P \subseteq R$, $P \in \overline{A^*}(M)$ if and only if P is the center of a Rees valuation of M. We also provide a number of other results concerning $\overline{A^*}(M)$, including an analogue of McAdam's theorem invoking the analytic spread and the fact that the primes in $\overline{A^*}(M)$ are induced from any faithfully flat extension of R. Furthermore, we show the important containment $\overline{A^*}(M) \subseteq A^*(M)$, where $A^*(M)$ denotes the stable value of $\mathrm{Ass}(F_n/M_n)$. These results are module analogues of well-known results for ideals, but are non-trivial extensions in that there is no obvious way to induct on the rank of M to deduce our results from the ideal case. Another problem one confronts in the module case is the following. Many of the results for ideals reduce to the principal case via the extended Rees ring of an ideal. And while there is a notion of Rees ring for M, there is nothing analogous to the extended Rees ring that would reduce the general case to something like a free module or cyclic module. Nevertheless, the Rees ring of M will play a vital role in our investigations, in that the essential prime divisors of the Rees ring of M act as intermediaries in proofs of our characterizations, much as they do in the ideal case.

We now describe the contents of this paper. We begin in section two by recalling a number of relevant definitions and constructions; we also give a few technical results needed for the rest of the paper. In section three, subsection one and two, we begin by describing the Rees valuations of M and prove a number of technical results that are used in the main results of that section. In Section 3.3 we present our characterizations for a prime P to belong to $\overline{A^*}(M)$. In Section 3.6 we use the results from Section 3.3 to prove that $\overline{A^*}(M)$ is contained in $A^*(M)$ and also that $\overline{A^*}(M)$ is contained in $\overline{A^*}(I_r(M))$, where $I_r(M)$ denotes the ideal of $r \times r$ minors of the matrix whose columns are the generators of M. The focus in section four is on applications to two and three dimensional local rings. For a two dimensional Cohen–Macaulay local ring or a three dimensional regular local ring, we show (with suitable hypothesis on M) that if the maximal ideal belongs to $\overline{A^*}(M)$, then one can give an explicit positive integer n_0, expressed in terms of invariants of R and M, such that the maximal ideal must be in the sets $\mathrm{Ass}(F_n/M_n)$ and $\mathrm{Ass}(F_n/\overline{M_n})$ for all $n \geqslant n_0$. The results extend to modules results that are known for ideals by various authors, including Huneke, McAdam, and Sally.

2. Preliminaries

In this section we will introduce some notational conventions and definitions as well as give some technical results which facilitate our work in subsequent sections. Throughout R will be a Noetherian, commutative ring. All modules will be finitely generated R-modules, unless stated otherwise. We work with a fixed R-module M contained in a finitely generated free module $F = R^r$. We write $I_r(M)$ to denote the ideal of $r \times r$ minors of the matrix whose columns generate M. For most of our results we assume $\mathrm{height}(I_r(M)) > 0$. In particular, this means that if R is a domain, then $\mathrm{rank}(M) = r$. There are two reasons for making this assumption. For an ideal $J \subseteq R$, this is what's required in order to have $\overline{A^*}(J)$ correspond to the centers of Rees valuations. The second reason is that it is highly desirable that the Rees ring of M and the symmetric algebra of F have the same quotient field. We begin by describing the powers of the modules we are interested in. As is the case with ideals, the powers in question can be described in terms of the graded components of a finitely generated R-algebra determined by the module.

2.1. The Rees ring

Fix a basis $e_1, \ldots, e_r$ of F, and let $\mathcal{F} = R[t_1, \ldots, t_r]$ with $t_1, \ldots, t_r$ indeterminates over R corresponding to the basis elements chosen. Note that $\mathcal{F}$ is just the symmetric algebra of F. Let $A = (a_{ij})$ be an $r \times m$ matrix whose columns (with respect to the given basis) generate M. For $1 \leqslant j \leqslant m$, let $\tilde{A}_j = \sum_{i=1}^{r} a_{ij} e_i$ be the jth column of A, and let $C_j = \sum_{i=1}^{r} a_{ij} t_i$ be the linear form in $\mathcal{F}$ corresponding to $\tilde{A}_j$. By abuse of terminology we define the *Rees ring* of M (with respect to the embedding of M into F) to be the subring of $\mathcal{F}$ generated over R by these linear forms. This will be denoted $\mathcal{R}_F(M)$, or simply $\mathcal{R}(M)$ or $\mathcal{R}$ if there is no question as to which modules we are referring to. Thus we have $\mathcal{R} = R[C_1, \ldots, C_m] \subseteq \mathcal{F}$. While there has been common agreement as to what the Rees algebra of a module M should be when R is a domain and M is torsion-free, there has not been a rigorous effort to describe a Rees algebra for arbitrary M until the recent paper [1]. Thus, while, strictly speaking, our ring $\mathcal{R}(M)$ is not always the Rees algebra of M as described in [1], it agrees with it in a number of important cases (e.g., when M has a rank). The point in [1] is that a true Rees algebra should not depend upon the embedding of M into F (or even require such an embedding), while we are interested in primes associated to powers of M that may depend upon the embedding, just as associated primes of an ideal (or its powers) depend on the embedding of the ideal into the ring.

The nth graded component of $\mathcal{R}$ will be denoted M_n. When M has a rank, i.e., there exists $l > 0$ such that for all $P \in \operatorname{Ass}(R)$, M_P is a free R_P-module of rank l, then M_n is easily seen to be the nth symmetric power of M, modulo its R-torsion. Thus, in this case, $\mathcal{R}(M)$ is just the symmetric algebra of M modulo its R-torsion. In any case, $\mathcal{R}(M)$ is certainly the image of the symmetric algebra of M in the symmetric algebra of F. Hence M_n is a submodule of F_n, where F_n is the nth graded component of $\mathcal{F}$, which is a free module of rank $\binom{n+r-1}{r-1}$. Thus M_n is the submodule of $F_n = R^{\binom{n+r-1}{r-1}}$ generated over R by the column vectors of A_n, where the columns of A_n are obtained by fixing an ordering on the monomials of degree n in $t_1, \ldots, t_r$ and reading off the coefficients of the monomials of degree n in all n-fold products of $C_1, \ldots, C_m$. To illustrate this construction, let M be the submodule of $F = R^2$ generated by the columns of

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then $C_1 = a_{11}t_1 + a_{21}t_2$ and $C_2 = a_{12}t_1 + a_{22}t_2$. Therefore,

$$\begin{aligned} C_1^2 &= a_{11}^2 t_1^2 + 2a_{11}a_{21}t_1t_2 + a_{21}^2 t_2^2, \\ C_1C_2 &= a_{11}a_{12}t_1^2 + (a_{11}a_{22} + a_{21}a_{12})t_1t_2 + a_{21}a_{22}t_2^2, \\ C_2^2 &= a_{12}^2 t_1^2 + 2a_{12}a_{22}t_1t_2 + a_{22}^2 t_2^2 \end{aligned}$$

are the 2-fold products of C_1 and C_2. Thus M_2 is the submodule of $F_2 = R^3$ generated by the columns of

$$A_2 = \begin{pmatrix} a_{11}^2 & a_{11}a_{12} & a_{12}^2 \\ 2a_{11}a_{21} & a_{11}a_{22} + a_{21}a_{12} & 2a_{12}a_{22} \\ a_{21}^2 & a_{21}a_{22} & a_{22}^2 \end{pmatrix}.$$

To continue describing our notation, let $f : R \to S$ be a homomorphism of Noetherian rings. Let $h : R^m \to F$ be the homomorphism corresponding to the matrix A whose image is M. Then the *extension* of M to S, denoted MS, is the image of the map $h \otimes_R S : R^m \otimes_R S \to F \otimes_R S \cong S^r$. This is the submodule of S^r generated by columns of the matrix A after applying f to the entries. Thus if $C_1, \ldots, C_m$ are the linear forms in $\mathcal{F}$ corresponding to the generators of M and $C'_1, \ldots, C'_m$ are the linear forms in $\mathcal{F} \otimes_R S$ after applying f to the coefficients, then $\mathcal{R}(MS) = S[C'_1, \ldots, C'_m]$. Hence $M_n S = (MS)_n$ for all $n \geqslant 1$. It also follows from the functorial properties of the tensor product that if $g : S \to T$ is another homomorphism with T a Noetherian ring, then $MT = (MS)T$. The *contraction* of $M_n S$ to F_n, denoted $M_n S \cap F_n$, is the set of elements f of F_n such that the image of f in $F_n S = F_n \otimes_R S$ is in $M_n S$. We will use this extension-contraction notation heavily throughout this paper. Here are some special cases we will often encounter. If $J \subseteq R$ is an ideal and $S = R/J$ then $MS = (M + JF)/JF \subseteq F/JF$ and we have

$$\mathcal{R}_{FS}(MS) = \frac{R}{J}[\overline{C_1}, \ldots, \overline{C_m}] \cong \frac{\mathcal{R}}{J\mathcal{F} \cap \mathcal{R}},$$

where C_i is the linear form in $\mathcal{F}$ corresponding to the ith column of A and $\overline{C_i}$ is the linear form in $(R/J)[t_1, \ldots, t_s]$ obtained from C_i by reducing the coefficients modulo J. If $P \subseteq R$ is a prime ideal and $S = R_P$, then $MR_P = M \otimes R_P = M_P \subseteq F_P$ by flatness. Furthermore,

$$\mathcal{R}(MR_P) = R_P[C_1, \ldots, C_m] \cong \mathcal{R}(M) \otimes_R R_P.$$

If (R, m) is local and $S = \hat{R}$ is the m-adic completion of R, then $M\hat{R} \cong \hat{M} \subseteq \hat{F}$ as $\hat{R}$ is a faithfully flat extension of R, and

$$\mathcal{R}(M\hat{R}) = \hat{R}[C_1, \ldots, C_m] \cong \mathcal{R}(M) \otimes_R \hat{R}.$$

A local ring (R, m) is said to be *quasi-unmixed* if $\dim(\hat{R}/q) = \dim(R)$ for every minimal prime ideal $q \in \operatorname{Spec}(\hat{R})$. A ring R is said to be *locally quasi-unmixed* if R_p is quasi-unmixed for all $p \in \operatorname{Spec}(R)$. If $A \subseteq B$ are domains then we will denote the *transcendence degree* of B over A by $\operatorname{trdeg}_A(B)$. It is well known that if A is a Noetherian domain, B is an extension ring of A which is a domain, and $P \in \operatorname{Spec}(B)$, then with $p = P \cap A$ we have

$$\operatorname{height}(P) + \operatorname{trdeg}_{A/p}(B/P) \leqslant \operatorname{height}(p) + \operatorname{trdeg}_A(B) \tag{2.1.1}$$

(see for instance [4, Theorem 15.5]). If a domain A satisfies the condition that the inequality in (2.1.1) is an equality for every finitely generated extension domain B of A, then A is said to *satisfy the dimension formula*. A Noetherian domain A satisfies the dimension formula if and only if A is locally quasi-unmixed [9, Theorem 3.6]. Therefore if A is a complete local domain then A satisfies the dimension formula, as complete local domains are clearly quasi-unmixed.

Remark 2.1.1. If R is a domain and M is a rank r submodule of $F = R^r$, then for any non-zero maximal minor δ of M, $\mathcal{R}_\delta = \mathcal{F}_\delta$. Thus the quotient field of $\mathcal{R}$ is the same as that of $\mathcal{F}$. Hence $\operatorname{trdeg}_R \mathcal{R} = r$.

The next proposition is quite useful for reducing to the case that R is a domain. It follows easily in standard fashion from the fact that $\mathcal{R}(M)$ is a subring of a polynomial ring over R.

Proposition 2.1.2. *The map $\phi : \operatorname{Spec}(R) \to \operatorname{Spec}(\mathcal{R})$ defined by $\phi(p) = p\mathcal{F} \cap \mathcal{R}$ is injective and order preserving. This map induces a bijection between the minimal prime ideals of R and the minimal prime ideals of $\mathcal{R}$. The same is true for the associated prime ideals of R and $\mathcal{R}$.*

It is worth pointing out that Proposition 2.1.2 holds if we replace $\mathcal{R}$ with $\mathcal{R}[t_i^{-1}]$ for some $1 \leqslant i \leqslant r$ using the correspondence $p \mapsto p\mathcal{F}[t_i^{-1}] \cap \mathcal{R}[t_i^{-1}]$. The proof is the same, noting that $\mathcal{F}[t_i^{-1}]$ is the localization of $\mathcal{F}$ at the multiplicatively closed set generated by t_i, and that extensions of prime or primary ideals of R to $\mathcal{F}[t_i^{-1}]$ are prime or primary and do not contain t_i.

Proposition 2.1.2 above and [13], Proposition 2.2 together yield:

Proposition 2.1.3. *Let $d = \dim R$ and M be a submodule of $F = R^r$. Then*

$$\dim \mathcal{R} = \max\left\{\dim\left(\frac{R}{p}\right) + \operatorname{rank}\left(\frac{M + pF}{pF}\right) \;\middle|\; p \in \operatorname{Ass}(R)\right\}.$$

Furthermore, if M has rank r then $\dim \mathcal{R} = d + r = d + \operatorname{height}(\mathcal{R}_+)$. Here $\mathcal{R}_+ = \bigoplus_{n=1}^{\infty} M_n$ is the irrelevant homogeneous ideal of $\mathcal{R}$.

2.2. Integral closure

We now consider the integral closure of M in F, and more generally, the integral closure of M_n in F_n. For this, we take the integral closure of $\mathcal{R}$ in $\mathcal{F}$. This is a graded subring of $\mathcal{F}$ (see for instance [14, Theorem 11]). Define the *integral closure of M_n in F_n*, denoted $\overline{M_n}$, to be the nth graded component of this ring, which is a submodule of F_n. If R is a domain then Rees, in [11], defines the integral closure M_n in F_n to be the set of elements x in F_n such that $x \in M_n V$ for all discrete valuation rings V between R and its fraction field. If R is not a domain Rees defines the integral closure of M_n in F to be the set of elements x of F_n such that the image of x in F_n/qF_n is in $\overline{(M_n + qF_n)/qF_n}$ for all minimal prime ideals q of R. Our definition agrees with the definition of the integral closure of a module given by Rees by Theorem 1.3 of [11] and Proposition 2.2.2 below. Note that $x \in F_n$ is in $\overline{M_n}$ if and only if x satisfies an equation of the form

$$x^l + m_1 x^{l-1} + \cdots + m_{l-1} x + m_l = 0$$

with $m_i \in M_{ni}$, where the sums and products occur in $\mathcal{F}$.

Remark 2.2.1. Let J be the ideal of $\mathcal{F}$ generated by $C_1, \ldots, C_m$, with $C_1, \ldots, C_m$ the linear forms in $\mathcal{F}$ corresponding to the generators of M. By degree considerations, for $x \in F_n$, we have $x \in M_n$ if and only if $x \in J^n$, and $x \in \overline{M_n}$ if and only if $x \in \overline{J^n}$. With these comments and those in the paragraph above, the proof of the next proposition is straight-forward.

Proposition 2.2.2. *Let R be a Noetherian ring and M a submodule of $F = R^r$. Then for all $n > 0$, $x \in F_n$ is in $\overline{M_n}$ if and only if $\tilde{x}$, the image of x in F_n/qF_n, is in $\overline{((M_n + qF_n)/qF_n)}$ for every minimal prime ideal q of R.*

The following lemma generalizes Lemma 3.15 from [5], which says that the integral closure of an ideal I of R is equal to the contraction to R of the integral closure of the extension of I to a faithfully flat extension of R.

Lemma 2.2.3. *Let R be a Noetherian ring and M a submodule of $F = R^r$. Let T be a Noetherian faithfully flat extension of R. Then $\overline{M_nT} \cap F_n = \overline{M_n}$. Moreover if $P \in \operatorname{Ass}(F_n/\overline{M_n})$ then there exists $Q \in \operatorname{Ass}(F_nT/\overline{M_nT})$ such that $Q \cap R = P$.*

Proof. Note that the Rees ring of MT is $\mathcal{R}' = \mathcal{R} \otimes T$, so that $M_nT = (MT)_n$. Let $\mathcal{F}' = \mathcal{F} \otimes T$ and $F_n' = F_n \otimes T$, which is the degree n component of $\mathcal{F}'$. Let J be as before. Restating Remark 2.2.1 gives

$$\overline{J^n} \cap F_n = \overline{M_n} \quad \text{and} \quad \overline{J^n\mathcal{F}'} \cap F_n' = \overline{(MT)_n}.$$

Thus we have

$$\overline{(MT)_n} \cap F_n = \left(\overline{J^n\mathcal{F}'} \cap F_n'\right) \cap F_n = \left(\overline{J^n\mathcal{F}'} \cap \mathcal{F}\right) \cap F_n.$$

By the ideal case this last module is $\overline{J^n\mathcal{F}} \cap F_n = \overline{M_n}$. The second statement now follows along similar lines, since associated primes of contracted modules or ideals lift over an extension of Noetherian rings. □

2.3. Free summands

In this section we deal with a technical matter encountered upon localization. Even if we begin with a local ring (R, m) and a module $M \subseteq mF$, if we localize at some prime Q different from m, it is often the case that $M_Q \nsubseteq QF_Q$. In this case a free R_Q summand splits from M_Q, and we want to discuss the effect this has on the objects under consideration. So we assume for this section that (R, m) is a Noetherian local ring and that $M \nsubseteq mF$. Then there exists a free submodule G of M, a free submodule H of F of rank t, and a submodule N of M such that $M = G \oplus N \subseteq G \oplus H = F$ and $N \subseteq mH$. Furthermore, given an element $f \in F\backslash(M + mF)$, we may choose H so that f is part of a basis for H. With this set-up, the following proposition is straightforward.

Proposition 2.3.1. *In the situation of described above, there exists a new set of variables $x_1, \ldots, x_r$ for $\mathcal{F}$ such that*

$$\mathcal{R}_F(M) \cong \mathcal{R}_H(N)[x_{t+1}, \ldots, x_r]$$

with $x_{t+1}, \ldots, x_r$ indeterminates over $\mathcal{R}_H(N)$. Furthermore, $\mathcal{R}_H(N)$ is generated over R by linear forms in the indeterminates $x_1, \ldots, x_t$ with coefficients in m.

Maintaining the notation above, let $\mathcal{G} = R[x_{t+1}, \ldots, x_r] \cong \operatorname{Sym}(G)$. Then Proposition 2.3.1 says that $\mathcal{R}_F(M) \cong \mathcal{R}_H(N) \otimes \mathcal{G}$. On the module level, this says that

$$M_n \cong \bigoplus_{i=0}^{n} (N_{n-i} \otimes G_i) \cong \bigoplus_{i=0}^{n} N_{n-i}^{\binom{i+l-1}{l-1}}$$

where $l = \text{rank}(G) = r - t$. Note that $N_0 = R$ so that $G_n = N_0^{\binom{n+l-1}{l-1}}$ is the nth summand. For example, if $\text{rank}(F) = 5$, $\text{rank}(G) = 3$, and $\text{rank}(H) = 2$ then

$$M_2 \cong N_2 \oplus (N \oplus N \oplus N) \oplus G_2 \quad \text{and} \quad M_3 \cong N_3 \oplus N_2^3 \oplus N^6 \oplus G_3.$$

Now we also have that

$$\overline{\mathcal{R}_F(M)} \cong \overline{\mathcal{R}_H(N)[x_{t+1}, \dots, x_r]} = \overline{\mathcal{R}_H(N)}[x_{t+1}, \dots, x_r].$$

Intersecting with $\mathcal{F}$ and comparing homogeneous components we see that

$$\overline{M_n} \cong \bigoplus_{i=0}^{n} \overline{N_{n-i}}^{\binom{i+l-1}{l-1}}.$$

Clearly the above direct sum decompositions are embedded into similar decompositions relating F_n, G_n and H_n. Thus we obtain

$$\frac{F_n}{M_n} \cong \bigoplus_{i=0}^{n-1} \left(\frac{H_{n-i}}{N_{n-i}}\right)^{\binom{i+l-1}{l-1}} \quad \text{and} \quad \frac{F_n}{\overline{M_n}} \cong \bigoplus_{i=0}^{n-1} \left(\frac{H_{n-i}}{\overline{N_{n-i}}}\right)^{\binom{i+l-1}{l-1}}. \tag{2.3.2}$$

2.4. *Reductions and analytic spread*

Let $N \subseteq M$ be a submodule. One says that N is a *reduction* of M (in F) if $\overline{N} = \overline{M}$ or equivalently if $\mathcal{R}(M)$ is integral over $\mathcal{R}(N)$. By the Artin–Rees lemma, this integrality is equivalent to saying that $N \cdot M_n = M_{n+1}$ for $n \gg 0$. A reduction N of M is a *minimal reduction* of M if it does not properly contain any other reduction of M. A detailed study of reductions was initiated by Rees in [11]. An easy, yet important fact is that free modules do not admit proper reductions. The following lemma gives the case that we will need.

Lemma 2.4.1. *Let R be a Noetherian ring. Let $M \subsetneq F = R^r$. Then M is not a reduction of F.*

Proof. Assume by way of contradiction that M is a reduction of F. By localizing at a prime in the support of F/M, we may assume that (R, m) is local. By our discussion in the previous section, we may write

$$M = G \oplus N \subseteq G \oplus H = F$$

with H and G free R-modules and $N \subseteq mH$. By Proposition 2.3.1,

$$\mathcal{R}_F(M) = \mathcal{R}_H(N)[x_{t+1}, \dots, x_r].$$

Note, that $t > 0$ since $M \neq F$. Now, by our hypothesis, $\mathcal{F}$ is integral over $\mathcal{R}_F(M)$. Therefore, x_1 is integral over $\mathcal{R}_F(M)$. Thus, in the notation of Remark 2.2.1, x_1 is integral over the ideal J in $\mathcal{F}$. In particular, some power of x_1 belongs to J. But this is a contradiction, since $N \subseteq mH$. Indeed, this latter condition implies that for every $f \in J$, every coefficient of a monomial involving x_1 belongs to m, and this precludes any power of x_1 belonging to J. □

Corollary 2.4.2. *For all $n \geqslant 1$, the supports of the modules F_n/M_n and $F_n/\overline{M_n}$ are the same and independent of $n \geqslant 1$.*

Proof. Let $P \subseteq R$ be a prime ideal. Clearly, if $(F/M)_P = 0$, then $(F_n/M_n)_P = (F_n/\overline{M_n})_P = 0$ for all n. Suppose now that $(F_n/M_n)_P = 0$, for some $n > 1$. Then, $(F_{n+1})_P = (M_nF_1)_P \subseteq (M_1F_n)_P$, so M_P is a reduction of F_P. By the previous lemma, $M_P = F_P$. Similarly, one can show that if $(F_n)_P = (\overline{M_n})_P$ for some n, then M_P is a reduction of F_P, so $M_P = F_P$. □

Let (R, m) be a local Noetherian ring and M a submodule of $F = R^r$. The ring $\mathcal{R}/m\mathcal{R}$ is called the *fiber ring* of M. The *analytic spread* of M is defined to be the dimension of the fiber ring, and will be denoted $l(M)$. Elements $a_1, \dots, a_s \in F$ are said to be *analytically independent in M* if whenever $f(X_1, \dots, X_s) \in R[X_1, \dots, X_s]$ is a homogeneous form of degree n such that $f(a_1, \dots, a_s) \in mM_n$, then all coefficients of f are in m. We say that $a_1, \dots, a_s \in F$ are *analytically independent* if whenever $f(X_1, \dots, X_s) \in R[X_1, \dots, X_s]$ is a homogeneous form of degree n such that $f(a_1, \dots, a_s) = 0$, then all coefficients of f are in m. It is straightforward to verify that $a_1, \dots, a_r \in F$ are analytically independent if and only if $a_1, \dots, a_r$ are analytically independent in the submodule of F that they generate. Note that it follows from this, that if $M \subseteq F$ is generated by m analytically independent elements then M_n is minimally generated by the monomials of degree n in the generators of M, in other words $\mu(M_n) = \binom{n+m-1}{m-1}$ for all $n \geqslant 1$.

The next proposition summarizes the basic facts concerning minimal reductions for modules. The statements and proofs are entirely analogous to the ideal case. See the discussion in [13] preceding Proposition 2.3 for details.

Proposition 2.4.3. *Let (R, m) be local with infinite residue field, and suppose that $M \subseteq F = R^r$. Then there exists a minimal reduction of M. Furthermore, if N is a reduction of M then the following hold*:

(i) *N is a minimal reduction of M if and only if $\mu(N) = l(M)$.*
(ii) *If N is a minimal reduction of M, then the elements of a minimal generating set for N are analytically independent.*
(iii) *If M is generated by analytically independent elements, then $m\mathcal{R}(M)$ is a prime ideal.*

Using the previous proposition, one easily proves the following lemma.

Lemma 2.4.4. *Let (R, m) be a local ring, let $M \subseteq F = R^r$, and let $P \in \operatorname{Spec}(R)$. Then $l(M) \geqslant l(M_P)$.*

We now state some bounds on the analytic spread of M which are given in [13, Proposition 2.3].

Proposition 2.4.5. *Let M be a submodule of $F = R^r$.*

(i) *If $\dim R > 0$, then $l(M) \leqslant \dim R + r - 1$.*
(ii) *If $\operatorname{height}(I_r(M)) > 0$, then $r \leqslant l(M) \leqslant \dim R + r - 1$.*

Proposition 2.4.6. *Let (R,m) be a local Noetherian ring and $M \subseteq F = R^r$. For any minimal prime ideal q of R, $l(\frac{M+qF}{qF}) \leqslant l(M)$ and if $\dim R > 0$, then equality holds for some minimal prime ideal q.*

2.5. Two extreme cases

In this subsection we want to record two extreme cases for a prime $P \subseteq R$ to be in $\overline{A^*}(M)$ or $A^*(M)$ as well as record an observation that will often allow us to assume that the depth of R is positive.

Proposition 2.5.1. *Let $P \subseteq R$ be a prime ideal such that $M_P \neq F_P$.*

(i) *If $P \in \mathrm{Ass}(R)$, then $P \in A^*(M)$.*
(ii) *If P is minimal in the support of F/M, then $P \in \overline{A^*}(M) \cap A^*(M)$.*

Proof. We may assume that R is a local ring with maximal ideal P. If $P \in \mathrm{Ass}(R)$, write $P = (0 :_R c)$, for some $c \in R$. As in Section 2.3, we write $F = H \oplus G$ and $M = N \oplus G$, where $N \subseteq PH$. Then as noted in Section 2.3, $H_n/N_n \subseteq F_n/M_n$ for all n. Take n_0 so that $c \notin P^n$, for $n \geqslant n_0$. Then for $n \geqslant n_0$ and any basis vector $v \in H_n$, $c \cdot v \notin N_n$, since $N_n \subseteq P^n H_n$. Since $P \cdot (c \cdot v) = 0$, we must have $P \in \mathrm{Ass}(H_n/N_n)$. Thus, $P \in A^*(M)$. Now assume P is minimal in the support of F/M. Then since $M_P \neq F_P$, the quotients $(F_n/M_n)_P$ and $(F_n/\overline{M_n})_P$ have non-zero finite length for all n, by Corollary 2.4.2. Thus, $P \in \overline{A^*}(M) \cap A^*(M)$. □

Proposition 2.5.2. *Let $L \subseteq R$ be a nilpotent ideal and set $S := R/L$. Then for a prime ideal $P \subseteq R$, $P \in \overline{A^*}(M)$ if and only if $PS \in \overline{A^*}(MS)$.*

Proof. First note that the discussion in Section 2.1 yields $(MS)_n = M_nS$ and $(FS)_n = F_nS$. Suppose $h \in F_n$ is such that its image in F_nS is integral over M_nS. If we let J denote the ideal in $\mathcal{F}$ generated by the linear forms in $\mathcal{F}$ determined by the generators of M, it follows from Remark 2.2.1 that the image of $\tilde{h}$ in $\mathcal{F} \otimes S$ is integral over the ideal $(J^n + L\mathcal{F})/L\mathcal{F}$. Here, $\tilde{h}$ denotes the form of degree n in $\mathcal{F}$ corresponding to h. Since L is a nilpotent ideal, it follows that $\tilde{h}$ is integral over J^n. Thus, h is integral over M_n. We also have $LF_n \subseteq \overline{M_n}$, so it follows that $\overline{M_n}S$ is the integral closure of M_nS in F_nS. The proposition follows immediately from this and a standard isomorphism theorem. □

3. Characterizations of asymptotic prime divisors

In this section we offer our main results that characterize the stable set of prime ideals associated to $F_n/\overline{M_n}$ for n large. Following McAdam in the case of ideals (see [5]), we refer to this finite set of prime ideals as the *asymptotic prime divisors of M*. Strictly speaking, this set of prime ideals depends upon the embedding of M in F, so a proper notation might reference F as well, but we opt to follow the convention already established for ideals. The existence of a finite set of asymptotic prime divisors for M is given by a theorem of Katz and Naude from [3]. This theorem says that if M is a submodule of $F = R^r$ then $\mathrm{Ass}(F_n/M_n) = \mathrm{Ass}(F_{n+1}/M_{n+1})$ and $\mathrm{Ass}(F_n/\overline{M_n}) = \mathrm{Ass}(F_{n+1}/\overline{M_{n+1}})$ for all $n \gg 0$. Their proof shows that the sets $\mathrm{Ass}(F_n/M_n)$ are increasing for $n \gg 0$ and the sets $\mathrm{Ass}(F_n/\overline{M_n})$ are increasing for all $n > 0$. Let $A^*(M)$ denote the stable value of $\mathrm{Ass}(F_n/M_n)$, and $\overline{A^*}(M)$ denote the stable value of $\mathrm{Ass}(F_n/\overline{M_n})$.

3.1. Rees valuations

In this subsection we will define the Rees valuations of M and mention Rees' result that these finitely many discrete valuations determine $\overline{M_n}$ for all n. Leading up to this we discuss the essential valuations of $\mathcal{R}(M)$. The initial part of this discussion has some overlap with section one in [11], but we include it because we have modified a number of things for our specific purposes.

For the time being, we assume that R is an integral domain with quotient field K and that M is a rank r submodule of $F = R^r$. Write $\mathcal{R} := \mathcal{R}_F(M)$ and let $\mathcal{K}$ be the quotient field of $\mathcal{R}$, so that $\mathcal{K}$ is also the quotient field of $\mathcal{F} = R[t_1, \dots, t_r]$. Note that the integral closure of $\mathcal{R}$ in $\mathcal{K}$, $\overline{\mathcal{R}}$, is a Krull domain [8, 33.10]. This means that there exists a defining family $\{\mathcal{V}_\lambda\}_{\lambda\in\Lambda}$ of discrete valuation rings of $\mathcal{K}$ such that $\overline{\mathcal{R}} = \bigcap_{\lambda\in\Lambda} \mathcal{V}_\lambda$, and, for all $0 \neq f \in \overline{\mathcal{R}}$, $f\mathcal{V}_\lambda \neq \mathcal{V}_\lambda$ for only finitely many λ. As is well known, $\{\overline{\mathcal{R}}_{\overline{\mathcal{P}}} \mid \overline{\mathcal{P}} \in \operatorname{Spec}(\overline{\mathcal{R}}),\ \operatorname{height}(\overline{\mathcal{P}}) = 1\}$ satisfies these conditions and is contained in any other defining family $\{\mathcal{V}_\lambda\}$. For this reason, the discrete valuation rings $\{\overline{\mathcal{R}}_{\overline{\mathcal{P}}} \mid \overline{\mathcal{P}} \in \operatorname{Spec}(\overline{\mathcal{R}}),\ \operatorname{height}(\overline{\mathcal{P}}) = 1\}$ are called the *essential valuations* of $\mathcal{R}$.

There is a finite subset of essential valuations of $\mathcal{R}$ that will be distinguished in the following. They are non-trivial in that they do not contain $\mathcal{F}$. They are the discrete valuations introduced by Rees in [11] and determine the integral closure of $\mathcal{R}$ in $\mathcal{F}$ and thus determine $\overline{M_n}$ for all n.

Lemma 3.1.1. *Let $\mathcal{V}$ be a discrete valuation ring between $\mathcal{R}$ and its quotient field. Then $M\mathcal{V} \neq F\mathcal{V}$ if and only if $I_r(M)\mathcal{V} \neq \mathcal{V}$.*

Proof. Since $\mathcal{V}$ is a principle ideal domain and $M\mathcal{V} \subseteq F\mathcal{V} = \mathcal{V}^r$, we have that $M\mathcal{V}$ is a free $\mathcal{V}$-module and there exists a basis $f_1, \dots, f_r$ of $F\mathcal{V}$ such that $M\mathcal{V} = \bigoplus_{i=1}^r y^{\alpha_i} f_i\mathcal{V}$, where $y \in \mathcal{V}$ is a uniformizing parameter for $\mathcal{V}$ and $\alpha_i \geqslant 0$ for $1 \leqslant i \leqslant r$. Now $I_r(M)\mathcal{V} = I_r(M\mathcal{V})$ is the zeroth Fitting ideal of $F\mathcal{V}/M\mathcal{V}$ with respect to the standard basis of $F\mathcal{V}$. On the other hand, $y^{\sum_{i=1}^r \alpha_i}\mathcal{V}$ is the zeroth Fitting ideal of $F\mathcal{V}/M\mathcal{V}$ with respect to the basis $f_1, \dots, f_r$ of $F\mathcal{V}$. Since the Fitting ideals are invariants of $F\mathcal{V}/M\mathcal{V}$, we must have $I_r(M)\mathcal{V} = y^{\sum_{i=1}^r \alpha_i}\mathcal{V}$. Now clearly $I_r(M)\mathcal{V} \neq \mathcal{V}$ if and only if $\alpha_j > 0$ for some $1 \leqslant j \leqslant r$ if and only if $M\mathcal{V} \neq F\mathcal{V}$. □

Definition 3.1.2. Observe that there are only a finite number of essential valuations $\mathcal{V}$ of $\mathcal{R}$ such that $I_r(M)\mathcal{V} \neq \mathcal{V}$. Hence, by Lemma 3.1.1, there are only a finite number of essential valuations $\mathcal{V}$ of $\mathcal{R}$ such that $M\mathcal{V} \neq F\mathcal{V}$, say $\mathcal{V}_1, \dots, \mathcal{V}_s$. Set $V_i := \mathcal{V}_i \cap K$. We call $V_1, \dots, V_s$ the *Rees valuations* of M.

The following observation plays a crucial role in any study of Rees valuations and is implicit in [11], section one. We state and prove it here for the convenience of the reader.

Observation 3.1.3. *Let R be a Noetherian domain, let K be the fraction field of R, and let $M \subseteq F = R^r$ be such that* $\operatorname{rank} M = r$. *Let (V, n) be a Rees valuation of M and let $\mathcal{V}$ be an essential valuation of $\mathcal{R} = \mathcal{R}(M)$ such that $M\mathcal{V} \neq F\mathcal{V}$ and $\mathcal{V} \cap K = V$. Set $W := \mathcal{R}(MV)_{n\mathcal{R}(MV)}$. Then $W = \mathcal{V}$. Thus, the Rees valuations of M are in one-to-one correspondence with the essential valuations $\mathcal{V}$ of $\mathcal{R}$ such that $M\mathcal{V} \neq F\mathcal{V}$.*

Proof. First note that since MV is a free V-module, $\mathcal{R}(MV)$ is an integrally closed polynomial ring. Thus, $n\mathcal{R}(MV)$ is a height one prime ideal, so W is a discrete valuation domain. Since $n\mathcal{R}(MV) \subseteq m_{\mathcal{V}} \cap \mathcal{R}(MV)$, if we show that $m_{\mathcal{V}} \cap \mathcal{R}(MV)$ has height one, then $n\mathcal{R}(MV) =$

$m_V \cap \mathcal{R}(MV)$, so that $W \subseteq \mathcal{V}$, and thus $W = \mathcal{V}$. Now, suppose $\mathcal{V} = \overline{\mathcal{R}}_{\mathcal{P}}$, with $\mathcal{P}$ a height one prime in $\overline{\mathcal{R}}$. Write $U := \overline{\mathcal{R}} \backslash \mathcal{P}$. Then, $\overline{\mathcal{R}} \subseteq \mathcal{R}(MV) \subseteq \overline{\mathcal{R}}_U$. Thus $[m_{\mathcal{V}} \cap \mathcal{R}(MV)]_U = m_{\mathcal{V}}$. This implies that height$(m_{\mathcal{V}} \cap \mathcal{R}(MV)) = 1$, which is what we want. □

The next proposition is a collection of facts most of which are due to Rees [11]. Some modifications of the statements and additions have been made to serve the purposes of this paper. In particular, we have added condition (iv) involving t_i^{-1}.

Proposition 3.1.4. *In the notation above, let $\mathcal{V}$ be a discrete valuation ring between $\mathcal{R}$ and its quotient field. Then the following are equivalent:*

(i) *$\mathcal{V}$ is an essential valuation of $\mathcal{R}$ and $I_r(M)\mathcal{V} \neq \mathcal{V}$.*
(ii) *$\mathcal{V}$ is an essential valuation of $\mathcal{R}$ and there exists i such that $t_i \notin \mathcal{V}$.*
(iii) *$\mathcal{V}$ is an essential valuation of $\mathcal{R}$ which is not an essential valuation of $\mathcal{F}$.*
(iv) *$\mathcal{V}$ is an essential valuation of $\mathcal{R}[t_i^{-1}]$ and $t_i^{-1}\mathcal{V} \neq \mathcal{V}$, for some $1 \leqslant i \leqslant r$.*

Proof. For (i) implies (ii), assume that $I_r(M)\mathcal{V} \neq \mathcal{V}$ and set $(V, n) := \mathcal{V} \cap K$. By Observation 3.1.3, $\mathcal{V} = \mathcal{R}(MV)_{n\mathcal{R}(MV)}$. If each t_i were in $\mathcal{V}$, then we would have $\mathcal{R}(MV) \subseteq V[t_1, \dots, t_r] \subseteq \mathcal{R}(MV)_{n\mathcal{R}(MV)}$. For any height one prime $Q \subseteq \mathcal{R}(MV)$ not equal to $n\mathcal{R}(MV)$, $Q \cap V = 0$, so

$$V[t_1, \dots, t_r] \subseteq K[t_1, \dots, t_r] \subseteq \mathcal{R}(MV)_Q.$$

It follows from this that $\mathcal{R}(MV) = V[t_1, \dots, t_r]$, so $MV = FV$, contrary to our assumption. Thus, $t_i \notin \mathcal{R}$, for some i.

The implication (ii) implies (iii) is obvious. For (iii) implies (iv), first note that $t_i \notin \mathcal{V}$ for some i, and hence $t_i^{-1} \in \mathcal{V}$ and $t_i^{-1}\mathcal{V} \neq \mathcal{V}$. Indeed, if all t_i belong to $\mathcal{V}$, then $\overline{\mathcal{R}} \subseteq \overline{R}[t_1, \dots, t_r] \subseteq \mathcal{V}$. Since $\mathcal{V}$ is an essential valuation of $\mathcal{R}$ there exists $\overline{\mathcal{P}}$ in Spec$(\overline{\mathcal{R}})$ such that $\overline{\mathcal{R}}_{\overline{\mathcal{P}}} = \mathcal{V}$. Then we have

$$\overline{\mathcal{R}}_{\overline{\mathcal{P}}} \subseteq \overline{R}[t_1, \dots, t_r]_{m_{\mathcal{V}} \cap \overline{R}[t_1, \dots, t_r]} \subseteq \mathcal{V}.$$

Thus, $\overline{\mathcal{R}}_{\overline{\mathcal{P}}} = \overline{R}[t_1, \dots, t_r]_{m_{\mathcal{V}} \cap \overline{R}[t_1, \dots, t_r]} = \mathcal{V}$. This contradicts that $\mathcal{V}$ is not an essential valuation of $R[t_1, \dots, t_r]$. Thus, $t_i \notin \mathcal{V}$ for some i. We now have $\overline{\mathcal{R}} \subseteq \overline{\mathcal{R}[t_i^{-1}]} \subseteq \mathcal{V}$. Thus $\overline{\mathcal{R}}_{\overline{\mathcal{P}}} \subseteq \overline{\mathcal{R}[t_i^{-1}]}_{m_{\mathcal{V}} \cap \overline{\mathcal{R}[t_i^{-1}]}} \subseteq \mathcal{V}$. But now this implies that $\overline{\mathcal{R}[t_i^{-1}]}_{m_{\mathcal{V}} \cap \overline{\mathcal{R}[t_i^{-1}]}} = \mathcal{V}$, so that $\mathcal{V}$ is an essential valuation of $\mathcal{R}[t_i^{-1}]$.

Finally, let $\mathcal{V}$ be as in (iv). We first show that $\mathcal{V}$ is an essential valuation of $\mathcal{R}$. Suppose $\mathcal{Q} \subseteq \overline{\mathcal{R}[t_i^{-1}]}$ is a height one prime such that $\overline{\mathcal{R}[t_i^{-1}]}_{\mathcal{Q}} = \mathcal{V}$. Set $\mathcal{P} := \mathcal{Q} \cap \mathcal{R}$. Then, since $t_i^{-1} \in \mathcal{Q}$, the transcendence degree of $\overline{\mathcal{R}[t_i^{-1}]}/\mathcal{Q}$ over $\mathcal{R}/\mathcal{P}$ is zero. Thus, by [5, Lemma 3.1], $\mathcal{Q} \cap \overline{\mathcal{R}}$ has height one. It follows immediately from this that $\overline{\mathcal{R}}_{\mathcal{Q} \cap \overline{\mathcal{R}}} = \mathcal{V}$, so $\mathcal{V}$ is an essential valuation of $\mathcal{R}$. To finish, we must show that $I_r(M)\mathcal{V} \neq \mathcal{V}$. Let $\delta \in I_r(M)$. Then $\delta t_i \in \mathcal{R} \subseteq \mathcal{V}$ as $\delta \in \operatorname{ann}(F/M) = \operatorname{ann}(\mathcal{F}_1/\mathcal{R}_1)$. As $t_i^{-1} \in m_{\mathcal{V}}$, this implies that $\delta = (\delta t_i)t_i^{-1} \in m_{\mathcal{V}}$. Thus $I_r(M)\mathcal{V} \subseteq m_{\mathcal{V}}$, as desired. □

We now state the general definition of Rees valuation.

Definition 3.1.5. Let R be a Noetherian ring and let M be a submodule of $F = R^r$ and assume $\operatorname{height}(I_r(M)) > 0$. Let $\{q_1, \ldots, q_u\}$ be the minimal prime ideals of R. For $i = 1, \ldots, u$, set $S_i := R/q_i$ and let $\{V_{ij}\}_{j=1}^{v_i}$ be the Rees valuations of MS_i. We will call the collection $\{V_{ij}\}$ the *Rees valuations* of M. Note that if for some $1 \leqslant i \leqslant u$, $MS_i = FS_i$, then we eliminate S_i from consideration.

The next theorem is essentially Theorem 1.7 in [11], though our notation is somewhat different. We record its statement for ease of reference. Recall our convention that $M_n V_{ij} \cap F_n$ means the set of elements in F_n that map to $M_n V_{ij}$ as a submodule of $F_n \otimes V_{ij}$.

Theorem 3.1.6 *(Rees). Let R be a Noetherian ring and M be a submodule of $F = R^r$. Suppose* $\operatorname{height}(I_r(M)) > 0$ *and let $\{V_{ij}\}$ as above denote the Rees valuations of M. Then for all $n \geqslant 1$,* $\overline{M_n} = \bigcap_{i=1}^{u}(\bigcap_{j=1}^{v_i}(M_n V_{ij} \cap F_n))$.

The last proposition in this subsection allows us to find a special linear form in $\mathcal{R}(M)$. This proposition will be used in a crucial way in the proofs of Theorem 3.3.3, Proposition 3.3.5, and Theorem 3.6.2.

Proposition 3.1.7. *Let R be a Noetherian domain and M be a rank r submodule of $F = R^r$. Let $\mathcal{V}$ be an essential valuation of $\mathcal{R}$ such that $M\mathcal{V} \neq F\mathcal{V}$. Set $\mathcal{P} := m_{\mathcal{V}} \cap \mathcal{R}$ and $P := m_{\mathcal{V}} \cap R$. Then there exists a linear combination f of $t_1, \ldots, t_r$ with coefficients in P such that $f \in \mathcal{R}$ and $f \notin \mathcal{P}$.*

Proof. We first reduce to the case that (R, P) is local. Note $\mathcal{V}$ is also an essential valuation of $\mathcal{R}(M_P)$ as the elements of $R \backslash P$ are units in $\mathcal{V}$ [4, Theorem 12.1]. Clearly $M_P\mathcal{V} = M\mathcal{V} \neq F\mathcal{V} = F_P\mathcal{V}$. Hence our conditions pass to the local situation. Now assume the result is true for a local ring. Let $g = \sum_{i=1}^{r}(a_i/s_i)t_i$ be an element of $\mathcal{R}(M_P)$ with coefficients in PR_P such that $g \notin \mathcal{P}\mathcal{R}(M_P) = m_{\mathcal{V}} \cap \mathcal{R}(M_P)$. Since $s_i \notin \mathcal{P}$ we may clear denominators to assume $s_i = 1$ for all i, while preserving the desired properties of g. Then clearly $a_i \in P$ and $f = \sum_{i=1}^{r} a_i t_i \notin \mathcal{P}$. So f is the desired element.

Now assume (R, P) is local. In this case, by our comments in Section 2.3, we may write $M = N \oplus G \subseteq H \oplus G = F$ with H and G free submodules of F of ranks $0 < t \leqslant r$ and $r - t$ respectively, and $N \subseteq PH$. Then, maintaining the notation of Proposition 2.3.1, $\mathcal{R} = \mathcal{R}_H(N)[x_{t+1}, \ldots, x_r]$, a polynomial ring over $\mathcal{R}_H(N)$. Set $\mathcal{P}_H := \mathcal{P} \cap \mathcal{R}_H(N)$. Suppose the conclusion of the proposition fails. Then $\mathcal{R}_H(N)_+ \subseteq \mathcal{P}_H$, i.e., $\mathcal{P}_H$ is an irrelevant ideal.

Now, as before, let $(V, n) := \mathcal{V} \cap K$, so that $\mathcal{R}(MV)_{n\mathcal{R}(MV)} = \mathcal{V}$. Note that we also have $\mathcal{R}(MV) = \mathcal{R}_{HV}(NV)[x_{t+1}, \ldots, x_r]$. Since $\mathcal{P}_H = n\mathcal{R}_{HV}(NV) \cap \mathcal{R}_H(N)$, it follows that $\mathcal{R}_{HV}(NV)_+ \subseteq n\mathcal{R}_{HV}(NV)$, and this is a contradiction. Indeed, $\mathcal{R}_{HV}(NV)$ is a polynomial ring in variables corresponding to linear forms in a minimal generating set for NV, so in fact *none* of these linear forms can belong to $n\mathcal{R}_{HV}(NV)$. This contradiction completes the proof of the proposition. □

3.2. Uniformly associated prime ideals

This subsection is entirely technical and consists of several lemmas and propositions that play a key role in our main results in Section 3.3. These results are based upon a number of known results, which we have refined in order to save extra information. This extra information will then tell us that a prime ideal associated to certain families of ideals can be written uniformly as a colon into members of the family through a single fixed element.

Lemma 3.2.1. *Let R be a Noetherian ring, $P \in \operatorname{Spec}(R)$, and $I, K \subseteq R$ be ideals with $I \subseteq K$. If there exists $c \in R_P$ such that $PR_P = \sqrt{JR_P :_{R_P} c}$ for all ideals J of R with $I \subseteq J \subseteq K$, then there exists $d \in R$ such that $P = \sqrt{J :_R d}$ for any ideal J of R with $I \subseteq J \subseteq K$.*

Proof. First choose k such that $P^k R_P \subseteq IR_P :_{R_P} c$. Then $(P^k R_P)c \subseteq IR_P$. Write $c = b/t$ with $b \in R$ and $t \in R \backslash P$. Then there exists $s \in R \backslash P$ such that $P^k bs \subseteq I$. Thus $P^k \subseteq (I :_R bs) \subseteq (J :_R bs) \subseteq (K :_R bs)$. Now let $y \in (K :_R bs)$. Then $(y/1)(b/t)s \in KR_P$. This implies that $(y/1)c \in KR_P$ as s is a unit in R_P. Thus $y/1 \in PR_P$ and so $y \in P$. So we have $P^k \subseteq (I :_R bs) \subseteq (J :_R bs) \subseteq (K :_R bs) \subseteq P$. Therefore $P = \sqrt{I :_R d} = \sqrt{J :_R d} = \sqrt{K :_R d}$ with $d = bs$. □

The following lemma is essentially Lemma 3.12 of [5]. McAdam shows that for P as in the lemma below, for large m, P is associated to every ideal J between I^m and its integral closure. We are merely saving some information from McAdam's proof. Namely, not only are we showing that P is associated to a collection of ideals J determined by I^m, but that uniformly, P is the radical of $(J : x)$ and x depends only on I.

Lemma 3.2.2. *Let R be a Noetherian ring, $I \subseteq R$ an ideal, and $q \in \operatorname{Spec}(R)$ a minimal prime ideal. If $P \in \operatorname{Spec}(R)$ is minimal over $I + q$, then there exists an $n \geqslant 1$ such that for any $m \geqslant n$, there exists $x \in R$ such that for any ideal J with $I^m \subseteq J \subseteq \overline{I^m}$ we have $P = \sqrt{J :_R x}$. Therefore $P \in \operatorname{Ass}(R/J)$ for all such J.*

Proof. By Lemma 3.2.1, and noting that $\overline{I^m R_P} = \overline{I^m} R_P$, we may localize at P to assume that (R, P) is local. In this case, as P is minimal over $I + q$, there exists $k \geqslant 1$ such that $P^k \subseteq I + q$. As q is minimal, there exists $x \notin q$ such that $q^n x = 0$ for all large n. Now Lemma 3.11 of [5] says that $\bigcap_{n \geqslant 1} \overline{I^n}$ is equal to the nilradical of R. Since $x \notin q$, x is also not in the nilradical of R. Hence if we choose n large enough we have that $x \notin \overline{I^n}$. Let $m \geqslant n$ and assume $I^m \subseteq J \subseteq \overline{I^m}$. Now $P^{2mk} \subseteq (I + q)^{2m} \subseteq I^m + q^m$, so that $P^{2mk} x \subseteq I^m x + q^m x = I^m x \subseteq I^m$. Thus as $x \notin \overline{I^m}$ and P is maximal, we have

$$P^{2mk} \subseteq \left(I^m :_R x\right) \subseteq (J :_R x) \subseteq \left(\overline{I^m} :_R x\right) \subseteq P.$$

Thus $P = \sqrt{J :_R x}$. □

We record the next lemma for ease of reference.

Lemma 3.2.3. *Let $R = \sum_{n=0}^{\infty} R_n$ be a graded Noetherian ring, and let J be a homogeneous ideal of R. Let $c = \sum_{i=m}^{n} c_i$ be any element of R, with $c_i \in R_i$. Then $\sqrt{J :_R c}$ is a homogeneous ideal and $\sqrt{J :_R c} = \sqrt{\bigcap_{i=m}^{n} (J :_R c_i)} = \bigcap_{i=m}^{n} \sqrt{J :_R c_i}$.*

Lemma 3.2.4. *Let $R = \sum_{n=0}^{\infty} R_n$ be a graded Noetherian ring. Let $P \in \operatorname{Spec}(R)$ be a homogeneous prime ideal, and let $I, K \subseteq R$ be homogeneous ideals with $I \subseteq K$. If there exists $c \in R$ such that $P = \sqrt{J :_R c}$ for all homogeneous ideals J with $I \subseteq J \subseteq K$, then there is a homogeneous element $d \in R$ such that $P = \sqrt{J :_R d}$ for all such ideals.*

Proof. Write $c = \sum_{i=0}^{n} c_i$ with $c_i \in R_i$ homogeneous. Then by assumption and Lemma 3.2.3 we have $P = \sqrt{K : c} = \bigcap_{i=1}^{n} \sqrt{K :_R c_i}$. Since P is prime we have $P = \sqrt{K :_R c_j}$ for some j. Now for all J as in the statement we have

$$P = \sqrt{J :_R c} = \bigcap_{i=1}^{n} \sqrt{J :_R c_i} \subseteq \sqrt{J :_R c_j} \subseteq \sqrt{K :_R c_j} = P.$$

Thus $P = \sqrt{J :_R c_j}$ for all homogeneous ideals J with $I \subseteq J \subseteq K$ as desired. □

Lemma 3.2.5. *Let R and T be Noetherian rings with T a flat extension of R. Let $Q \in \operatorname{Spec}(T)$ and $P \in \operatorname{Spec}(R)$ such that $P = Q \cap R$. Let $I \subseteq R$ be an ideal. If there exists $c \in T$ such that $Q = \sqrt{IT :_T c} = \sqrt{\overline{IT} :_T c}$, then there exists $d \in R$ such that $P = \sqrt{I :_R d} = \sqrt{\bar{I} :_R d}$.*

Proof. First note that T_P is a flat extension of R_P and $QT_P \cap R_P = PR_P$. Furthermore, if $I \subset R$ is an ideal satisfying $Q = \sqrt{IT :_T c} = \sqrt{\overline{IT} :_T c}$ for some $c \in T$, then $QT_P = \sqrt{IT_P :_{T_P} c} = \sqrt{\overline{IT_P} :_{T_P} c}$, using that integral closure commutes with localization. Hence by Lemma 3.2.1, it is enough to show the result when (R, P) is local, taking $\bar{I}$ for K in that lemma. Now choose k such that $Q^k \subseteq (IT :_T c)$. Then $c \in (IT :_T Q^k) \subseteq (IT :_T P^kT)$ as $P^kT \subseteq Q^k$. Now since Q is a proper ideal, we have $c \notin \overline{IT}$ and so $c \notin \bar{I}T$ as $\bar{I}T \subseteq \overline{IT}$. Since T is a flat extension of R, $(IT :_T P^kT) = (I :_R P^k)T$. So we have that $(I :_R P^k)T$ is not contained in $\bar{I}T$. Thus $(I :_R P^k)$ is not contained in $\bar{I}$. Let $d \in (I :_R P^k)\backslash\bar{I}$. Then $P^k \subseteq (I :_R d) \subseteq (\bar{I} :_R d) \subseteq P$ as P is maximal and $d \notin \bar{I}$. Therefore $P = \sqrt{I :_R d} = \sqrt{\bar{I} :_R d}$. □

Proposition 3.2.6. *Let R be a Noetherian ring and $x \in R$ a non-zerodivisor. Let $P \subseteq R$ be a prime ideal and suppose $P \in \overline{A^*}(xR)$. Then there exists $n \geqslant 1$ and $d \in R$ such that $P = \sqrt{(x^nR : d)} = \sqrt{(\overline{x^nR} : d)}$.*

Proof. Suppose we could prove the result over R_P. Then, there would exist $n \geqslant 1$ and $c \in R_P$ with $PR_P = \sqrt{(J : c)}$, for all ideals J in R_P, with $x^nR_P \subseteq J \subseteq \overline{x^nR_P}$. Then, what we wish to show would follow from Lemma 3.2.1. Thus, we may assume that R is local at P. Let $\hat{R}$ denote the P-adic completion of R. Then $\hat{P} \in \overline{A^*}(x\hat{R})$, so there exists a minimal prime $q \subseteq \hat{R}$ such that, for $S := \hat{R}/q$, $PS \in \overline{A^*}(xS)$. Since S is a quasi-unmixed local domain, height$(PS) = 1$, by [9, Theorem 3.8]. Thus, $P\hat{R}$ is minimal over $x\hat{R} + q$. If we now apply Lemma 3.2.2 followed by Lemma 3.2.5, we get what we want. □

Corollary 3.2.7. *Let R be a Noetherian domain, $P \in \operatorname{Spec} R$, and $0 \neq x \in P$. If there exists $Q \in \operatorname{Spec}(\overline{R})$ such that* height$(Q) = 1$ *and $Q \cap R = P$, then there exists $n \geqslant 1$ and $d \in R$ such that $P = \sqrt{x^nR :_R d} = \sqrt{\overline{x^nR} :_R d}$.*

Proof. By [5, Proposition 3.5], $P \in \overline{A}^*(xR)$. Now apply Proposition 3.2.6. □

3.3. The centers of Rees valuations

In this subsection we prove one of the main results of this paper, namely that the prime ideals in $\overline{A^*}(M)$ are exactly the centers of the Rees valuations of M. This is the module analogue of what is known in the ideal case. We begin with a crucial test for a prime ideal $P \subseteq R$ to belong to $\overline{A^*}(M)$.

Proposition 3.3.1. *Let R be a Noetherian ring and M be a submodule of $F = R^r$. Assume* $\operatorname{height}(I_r(M)) > 0$ *and let* $P \in \operatorname{Spec}(R)$. *Assume further that there exists* $\mathcal{P} \in \operatorname{Spec}(\mathcal{R})$ *satisfying the following conditions*:

(i) $\mathcal{P} \cap R = P$,
(ii) *there exists $f \in \mathcal{R}_1$ with coefficients in P such that $f \notin \mathcal{P}$,*
(iii) *there exists a non-zerodivisor $x \in P$ and a homogeneous element $d \in \mathcal{R}_l$ such that* $\mathcal{P} = \sqrt{x\mathcal{R} :_{\mathcal{R}} d} = \sqrt{\overline{x\mathcal{R}} :_{\mathcal{R}} d}$.

Then $P \in \overline{A^*}(M)$.

Proof. First localize R at P to assume (R, P) is local. Now choose n such that $\mathcal{P}^k \subseteq (x\mathcal{R} :_{\mathcal{R}} d)$ for all $k \geqslant n$. In particular, since $P \subseteq \mathcal{P}$ we have $P^k d \subseteq x\mathcal{R} \subseteq x\mathcal{F}$. Since all coefficients of f^k are in P^k, we have that $f^k d$ is divisible by x in $\mathcal{F}$. Say $f^k d = xq$ for some $q \in F_{k+l}$. Then we have that $\mathcal{P} = \sqrt{\overline{x\mathcal{R}} :_{\mathcal{R}} f^k d}$ as $f \notin \mathcal{P}$. It follows that

$$\mathcal{P} = \sqrt{\left(\overline{x\mathcal{R}} :_{\mathcal{R}} f^k d\right)} = \sqrt{(x\overline{\mathcal{R}} \cap \mathcal{R} :_{\mathcal{R}} xq)} = \sqrt{(x\overline{\mathcal{R}} :_{\overline{\mathcal{R}}} xq) \cap \mathcal{R}} = \sqrt{(\overline{\mathcal{R}} :_{\overline{\mathcal{R}}} q) \cap \mathcal{R}},$$

as x is a non-zerodivisor. Note also that $q \notin \overline{\mathcal{R}}$ as $\mathcal{P}$ is a proper ideal, so that $q \notin \overline{\mathcal{R}} \cap F_{k+l} = \overline{M_{k+l}}$. Choose m such that $\mathcal{P}^m \subseteq (\overline{x\mathcal{R}} :_{\mathcal{R}} f^k d)$. Then we have that $P^m \subseteq \mathcal{P}^m \subseteq (\overline{\mathcal{R}} :_{\overline{\mathcal{R}}} q) \cap \mathcal{R}$ and $q \in F_{k+l} \backslash \overline{M_{k+l}}$. Since the elements of P are homogeneous of degree zero when considered as elements of $\overline{\mathcal{R}}$ and q is homogeneous of degree $l + k$, this means that $P^m q \subseteq \overline{\mathcal{R}} \cap F_{k+l} = \overline{M_{k+l}}$. Since P is maximal and $q \in F_{k+l} \backslash \overline{M_{k+l}}$, we have $P \in \operatorname{Ass}(F_{k+l}/\overline{M_{k+l}})$. This holds for all $k \geqslant n$, so $P \in \overline{A^*}(M)$. □

Remark 3.3.2. Note that the assumption that $\mathcal{P} = \sqrt{x\mathcal{R} :_{\mathcal{R}} d} = \sqrt{\overline{x\mathcal{R}} :_{\mathcal{R}} d}$ is needed to ensure that the element q as chosen in the first paragraph of the proof of Proposition 3.3.1 is in $\mathcal{F}$. If we merely assume that $\mathcal{P} = \sqrt{\overline{x\mathcal{R}} :_{\mathcal{R}} d}$, then q will be in $\overline{\mathcal{F}}$ but it is not clear that q must be in $\mathcal{F}$. We will use the uniformity results of the previous subsection to write the centers on $\mathcal{R}$ of the essential valuations of $\mathcal{R}$ in the form $\mathcal{P} = \sqrt{x\mathcal{R} :_{\mathcal{R}} d} = \sqrt{\overline{x\mathcal{R}} :_{\mathcal{R}} d}$.

The following theorem is one of the main results of this paper. It shows, on the one hand, that the primes in $\overline{A^*}(M)$ are the centers of the Rees valuations of M, while, on the other hand, these primes are contractions from $\mathcal{R}$ of primes associated to the integral closure of powers of a principal ideal, which is reminiscent of the case for ideals (see [5]).

Theorem 3.3.3. *Let R be a Noetherian domain, M be a submodule of $F = R^r$ having rank r. Let P be a prime ideal of R. The following are equivalent*:

(i) $P \in \overline{A^*}(M)$.
(ii) *P is the center of a Rees valuation of M on R.*

(iii) *P contains $I_r(M)$ and there exists $0 \neq x \in R$ together with a prime ideal $\mathcal{P} \in \overline{A^*}(x\mathcal{R})$ so that $\mathcal{P} \cap R = P$.*

Proof. Without loss of generality we may assume that R is local with maximal ideal P. Assume $P \in \operatorname{Ass}(F_n/\overline{M_n})$, and write $P = (\overline{M_n} :_R c)$ with $c \in F_n \backslash \overline{M_n}$. Then by Theorem 3.1.6, $c \notin M_n V$ for some Rees valuation V of M. Now in a similar fashion to what was done in the proof of Lemma 3.1.1, let $f_1, \ldots, f_r$ be a basis of FV such that $MV = \bigoplus_{i=1}^r y^{\alpha_i} f_i V \subseteq \bigoplus_{i=1}^r f_i V = FV$, where $y \in V$ is a uniformizing parameter. For $1 \leqslant i_1 \leqslant \cdots \leqslant i_n \leqslant r$, let $f_{i_1,\ldots,i_n}$ denote the basis element of V_n corresponding to the product $f_{i_1} \cdots f_{i_n}$. Then we have

$$M_n V = \bigoplus_{1 \leqslant i_l \leqslant \cdots \leqslant i_n \leqslant r} y^{\alpha_{i_1} + \cdots + \alpha_{i_n}} f_{i_1,\ldots,i_n} V \subseteq F_n V.$$

Write $c = \sum_{1 \leqslant i_l \leqslant \cdots \leqslant i_n \leqslant r} c_{i_1,\ldots,i_n} f_{i_1,\ldots,i_n}$ with $c_{i_1,\ldots,i_n} \in V$. Let $v : K \to \mathbb{Z}$ denote the value function of V. Since $c \notin M_n V$, there exist $1 \leqslant k_1 \leqslant \cdots \leqslant k_n \leqslant r$ such that $v(c_{k_1,\ldots,k_n}) < \alpha_{k_1} + \cdots + \alpha_{k_n}$. However $Pc \subseteq \overline{M_n} \subseteq M_n V$, so $Pc_{k_1,\ldots,k_n}$ is contained in $y^{\alpha_{k_1} + \cdots + \alpha_{k_n}} V$. Thus $v(P) \geqslant 1$ and hence $P \subseteq m_V$. Therefore $P = m_V \cap R$ as P is the maximal ideal of R. It follows that P is the center of a Rees valuation, so (i) implies (ii).

Now, suppose that P is the center of the Rees valuation V on R. Then $V = \mathcal{V} \cap R$, where $\mathcal{V} = \overline{\mathcal{R}}_{\overline{\mathcal{P}}}$, for a height one prime $\overline{\mathcal{P}} \subseteq \overline{\mathcal{R}}$ and $I_r(M)\mathcal{V} \neq \mathcal{V}$. Let $\mathcal{P} := \overline{\mathcal{P}} \cap \mathcal{R}$. Then $\mathcal{P} \cap R = P$ and $I_r(M) \subseteq P$. Now let $0 \neq x \in P$. By the ideal case (see [5, Proposition 3.5]) we have that $\mathcal{P} \in \overline{A^*}(x\mathcal{R})$, and thus (ii) implies (iii).

Finally, suppose (iii) holds. Since $\mathcal{P} \in \overline{A^*}(x\mathcal{R})$, by Corollary 3.2.7, there exists $n \geqslant 1$ and $d \in \mathcal{R}$ such that $\mathcal{P} = \sqrt{x^n\mathcal{R} :_{\mathcal{R}} d} = \sqrt{\overline{x^n\mathcal{R}} :_{\mathcal{R}} d}$. By Lemma 3.2.4 we may also assume that d is a homogeneous element of $\mathcal{R}$, say of degree l. On the other hand, by the ideal case, $\mathcal{P}$ is the center of an essential valuation $\mathcal{V}$ of $\mathcal{R}$. Since $I_r(M) \subseteq P$, $I_r(M)\mathcal{V} \neq \mathcal{V}$, by Lemma 3.1.1. Now choose $f \in \mathcal{R}_1$ according to Proposition 3.1.7, i.e., f has its coefficients in P and $f \notin \mathcal{P}$. All of the conditions in Proposition 3.3.1 are satisfied and therefore $P \in \overline{A^*}(M)$. □

Remark 3.3.4. Maintain the notation in Theorem 3.3.3. The proof above shows that for a prime $\mathcal{P} \subseteq \mathcal{R}$, the following statements are equivalent:

(a) $\mathcal{P} \cap R \in \overline{A^*}(M)$.
(b) $I_r(M) \subseteq \mathcal{P}$ and $\mathcal{P} \in \overline{A^*}(x\mathcal{R})$, for some (any) $0 \neq x \in R \cap \mathcal{P}$.
(c) $\mathcal{P}$ is the center of an essential valuation $\mathcal{V}$ of $\mathcal{R}$ for which $I_r(M)\mathcal{V} \neq \mathcal{V}$.

Theorem 3.3.3 is true without the assumption that R is a domain. It will follow immediately from the domain case and Proposition 3.3.5. However, we need the domain case of Theorem 3.3.3 to prove Proposition 3.3.5, and so had to prove it first. We will state and prove the general result after Proposition 3.3.5.

Proposition 3.3.5. *Let R be a Noetherian ring and M be a submodule of $F = R^r$. Suppose* $\operatorname{height}(I_r(M)) > 0$ *and let $P \subseteq R$ be a prime ideal in the support of F/M. Then $P \in \overline{A^*}(M)$ if and only if there exists a minimal prime ideal q such that $P/q \in \overline{A^*}(\frac{M+qF}{qF})$.*

Proof. Without loss of generality we may localize at P to assume R is local with maximal ideal P. If $P \in \overline{A^*}(M)$, then we proceed as in the ideal case. Choose $n \gg 0$ such that

$\mathrm{Ass}(F_n/\overline{M_n})$ and write $P = (\overline{M_n} :_R f)$ with $f \in F_n \backslash \overline{M_n}$. Since $f \notin \overline{M_n}$, Proposition 2.2.2 says that there exists a minimal prime ideal q such that if we write $S := R/q$, the image of f in F_nS, say $\tilde{f}$, is not in $\overline{M_nS}$. Clearly $PS \cdot \tilde{f} \subseteq \overline{M_nS}$. Since PS is maximal and $\tilde{f} \notin \overline{M_nS}$, $PS = (\overline{M_nS} :_S \tilde{f})$. Increasing n if necessary gives $PS \in \overline{A^*}(MS)$, which is what we want.

Next assume that q is a minimal prime ideal of R and that $PS \in \overline{A^*}(MS)$, where $S := R/q$. Note that since P is in the support of M, $I_r(M) \subseteq P$, and it follows that $\dim(R) > 0$. We first note that we may assume that the depth of R is positive. Suppose that the depth of R is zero. Let $L := (0 : P^n)$, where n is chosen large enough so that R/L has positive depth. Note that L is a nilpotent ideal. Since q/L is a minimal prime of R/L, if we know the result when the depth of R is positive, then $P/L \in \overline{A^*}((M + LF)/LF)$. By Proposition 2.5.2, $P \in \overline{A^*}(M)$, which is what we want. Thus, we replace R/L by R and begin again assuming that R has positive depth.

To continue, we have $PS \in \overline{A^*}(MS)$, so by Theorem 3.3.3 and Remark 3.3.4 there exists a prime ideal $\mathcal{P}_S \subseteq \mathcal{R}(MS)$ so that $PS = \mathcal{P}_S \cap S$ and $\mathcal{P}_S$ is the center of an essential valuation $\mathcal{V}$ of $\mathcal{R}(MS)$ satisfying $I_r(M)\mathcal{V} \neq \mathcal{V}$. By Proposition 3.1.7, there exists an element $\tilde{f} \in \mathcal{R}(MS)_1$ such that the coefficients of $\tilde{f}$ belong to PS and $\tilde{f} \notin \mathcal{P}_S$. Let $\mathcal{P} \subseteq \mathcal{R}$ be the prime ideal corresponding to $\mathcal{P}_S$ and let $f \in \mathcal{R}$ be a preimage of $\tilde{f}$ such that $f \in \mathcal{R}_1$, f has coefficients in P and $f \notin \mathcal{P}$. Now let $x \in R$ be a non-zerodivisor. Then $\mathcal{P}_S \in \overline{A^*}(x\mathcal{R}(MS))$, by Remark 3.3.4. By the ideal version of this proposition (see [5, Proposition 3.18]), $\mathcal{P} \in \overline{A^*}(x\mathcal{R})$. Thus, by Proposition 3.2.6, there exists $n \geqslant 1$ and $d \in \mathcal{R}$ so that

$$\mathcal{P} = \sqrt{\left(x^n\mathcal{R} : d\right)} = \sqrt{\left(\overline{x^n\mathcal{R}} : d\right)},$$

and by Lemma 3.2.4, we may assume d is homogeneous. By Proposition 3.3.1, $P \in \overline{A^*}(M)$, which completes the proof. □

We will now state and prove Theorem 3.3.3 without the assumption that R is a domain.

Theorem 3.3.6. *Let R be a Noetherian ring, M be a submodule of $F = R^r$ such that* $\mathrm{height}(I_r(M)) > 0$. *For a P be a prime ideal of R, the following statements are equivalent*:

- (i) $P \in \overline{A^*}(M)$.
- (ii) *P is the center of a Rees valuation of M on R.*
- (iii) *P contains $I_r(M)$ and there exists $x \in R$ with* $\mathrm{height}(xR) > 0$ *and a prime ideal $\mathcal{P}$ in $\overline{A^*}(M)(x\mathcal{R})$ with $\mathcal{P} \cap R = P$. If* $\mathrm{grade}(P) > 0$, *we may take x to be a* (*any*) *non-zerodivisor in P.*

Proof. First note that each of the conditions imply that $I_r(M) \subseteq P$, so if P satisfies any of the conditions, P is in the support of F/M. Now suppose that (i) holds. By Proposition 3.3.5, there exists a minimal prime $q \subseteq P$ such that if we write $S := R/q$, $PS \in \overline{A^*}(MS)$. By Theorem 3.3.3, PS is the center of a Rees valuation V of MS on S. Clearly V has center P on R and by definition, V is a Rees valuation of M.

If (ii) holds, then by definition, there exists a minimal prime $q \subseteq P$ such that writing $S := R/q$, P is the center of a essential valuation $\mathcal{V}$ of $\mathcal{R}(MS)$ for which $I_r(M)\mathcal{V} \neq \mathcal{V}$. Since P has positive height, take $x \in P$ not in any minimal prime of R. Then if $\mathcal{P}_S$ denotes the center of $\mathcal{V}$ on $\mathcal{R}(MS)$, by Remark 3.3.4 $\mathcal{P}_S$ belongs to $\overline{A^*}(x\mathcal{R}(MS))$. Writing $\mathcal{P}$ for the preimage of $\mathcal{P}_S$ in $\mathcal{R}$, it follows from Proposition 3.18 in [5] that $P \in \overline{A^*}(x\mathcal{R})$. Thus, the first statement in (iii) holds. The second statement is clear.

Finally, if (iii) holds, then by [5, Proposition 3.18], there exists a minimal prime $\mathcal{Q} \subseteq \mathcal{P}$ such that $\mathcal{P}/\mathcal{Q} \in \overline{A^*}(x \cdot \mathcal{R}/\mathcal{Q})$. Thus, there exists a minimal prime $q \subseteq R$ such that if we write $S := R/q$ and $\mathcal{P}_S := \mathcal{P}/\mathcal{Q}$, $\mathcal{Q} = q\mathcal{F} \cap \mathcal{R}$, $\mathcal{R}/\mathcal{Q} = \mathcal{R}(MS)$ and $\mathcal{P}_S \in \overline{A^*}(x\mathcal{R}(MS))$. By Theorem 3.3.3, $PS \in \overline{A^*}(MS)$. Therefore, $P \in \overline{A^*}(M)$, by Proposition 3.3.5. Thus, (iii) implies (i) and the proof is complete. □

3.4. *Asymptotic primes via faithfully flat extensions*

In this section we note the important fact that the asymptotic primes of M are induced from any faithfully flat extension of R. In particular, when R is a local ring, the asymptotic primes of M lift to those of $\hat{M}$ and those of $\hat{M}$ contract back to those of M. Though this is certainly not unexpected, it requires work, just as in the ideal case.

We begin with a result that is similar in spirit to the case for ideals, in that it brings into play extensions of $\mathcal{R}(M)$ that look like extended Rees algebras. Unfortunately, unlike the case for ideals, the zeroth graded pieces of these rings are rather complicated and are certainly not just M in degree zero.

Proposition 3.4.1. *Let R be a Noetherian ring and M be a submodule of $F = R^r$. Assume* $\operatorname{height}(I_r(M)) > 0$ *and let $P \subseteq R$ be a prime ideal. Then $P \in \overline{A^*}(M)$ if and only if there exists $i = 1, \ldots, r$ and $\mathcal{P} \in \overline{A^*}(t_i^{-1}\mathcal{R}[t_i^{-1}])$ such that $\mathcal{P} \cap R = P$.*

Proof. We will first prove the proposition in the case that R is a domain. By Theorem 3.3.3 and the definition of Rees valuation, we have that $P \in \overline{A^*}(M)$ if and only if there is an essential valuation $\mathcal{V}$ of $\mathcal{R}$ such that $M\mathcal{V} \neq F\mathcal{V}$ and $m_{\mathcal{V}} \cap R = P$. Using Proposition 3.1.4, this holds if and only if for some i, there is an essential valuation $\mathcal{V}$ of $\mathcal{R}[t_i^{-1}]$ such that $t_i^{-1} \in m_{\mathcal{V}}$ and $m_{\mathcal{V}} \cap R = P$. On the other hand, combining [5, Lemma 3.2 and Proposition 3.5], we have that $\mathcal{P} \in \overline{A^*}(t_i^{-1}\mathcal{R}[t^{-1}])$ if and only if $\mathcal{P}$ is the center of an essential valuation of $\mathcal{R}[t_i^{-1}]$ such that $t_i^{-1} \in m_{\mathcal{V}}$, which completes the proof in this case R is a domain.

Now remove the assumption that R is a domain and assume that $P \in \overline{A^*}(M)$. Then there exists a minimal prime $q \subseteq P$ such that for $S := R/q$, $PS \in \overline{A^*}(MS)$, by Proposition 3.3.5. By the domain case, there exists $1 \leqslant i \leqslant r$ and a prime ideal $\mathcal{P}_S$ in $\overline{A^*}(t_i^{-1}\mathcal{R}(MS)[t_i^{-1}])$ such that $\mathcal{P}_S \cap S = PS$. Now, $\mathcal{R}(MS)[t_i^{-1}] = \mathcal{R}(M)[t_i^{-1}]/Q$, where $Q = q\mathcal{F}[t_i^{-1}] \cap \mathcal{R}(M)[t_i^{-1}]$, and Q is a minimal prime ideal in $\mathcal{R}(M)[t_i^{-1}]$. Let $\mathcal{P}$ be a prime ideal in $\mathcal{R}(M)[t_i^{-1}]$ such that $\mathcal{P}/Q = \mathcal{P}_S$. Then, $\mathcal{P}/Q \in \overline{A^*}(t_i^{-1}\mathcal{R}(MS)[t_i^{-1}])$. Hence $\mathcal{P} \in \overline{A^*}(t_i^{-1}\mathcal{R}(M)[t_i^{-1}])$, by the ideal case of Proposition 3.3.5 (see [5, Proposition 3.18]). Clearly $\mathcal{P} \cap R = P$.

Conversely assume that $\mathcal{P} \in \overline{A^*}(t_i^{-1}\mathcal{R}(M)[t_i^{-1}])$ and $\mathcal{P} \cap R = P$. Then there exists a minimal prime $Q \subseteq \mathcal{R}[t_i^{-1}]$ such that $\mathcal{P}/Q \in \overline{A^*}((t_i^{-1}\mathcal{R}(M)[t_i^{-1}] + Q)/Q)$. Say $Q = q\mathcal{F}[t_i^{-1}] \cap \mathcal{R}(M)[t_i^{-1}]$ with q a minimal prime ideal in R. Then for $S := R/q$, $PS = (\mathcal{P}/Q) \cap S$, so $PS \in \overline{A^*}(MS)$ by the domain case. Thus $P \in \overline{A^*}(M)$ by Proposition 3.3.5. □

Theorem 3.4.2. *Let R be a Noetherian ring, M a submodule of $F = R^r$ and assume* $\operatorname{height}(I_r(M)) > 0$. *Let T be a Noetherian ring that is a faithfully flat extension of R. For a prime $P \subseteq R$, $P \in \overline{A^*}(M)$ if and only if there exists a prime ideal $Q \subseteq T$ such that $Q \cap R = P$ and $Q \in \overline{A^*}(MT)$. In particular, if R is a local ring, then the $P \in \overline{A^*}(M)$ if and only if there exists a prime $Q \in \hat{R}$ such that $P = Q \cap R$ and $Q \in \overline{A^*}(M\hat{R})$.*

Proof. If $P \in \overline{A^*}(M)$, then such a Q exists by Lemma 2.2.3. Conversely, suppose that $Q \subseteq T$ is a prime ideal belonging to $\overline{A^*}(MT)$ and set $P := Q \cap R$. Then there exists $1 \leqslant i \leqslant r$ and $\mathcal{Q} \in \overline{A^*}(t_i^{-1}\mathcal{R}_{FT}(MT)[t_i^{-1}])$ such that $\mathcal{Q} \cap T = Q$, by Proposition 3.4.1. Since $\mathcal{R}_{FT}(MT)$ is faithfully flat over $\mathcal{R}(M)$, by the ideal case, $\mathcal{P} := \mathcal{Q} \cap \mathcal{R}(M)$ belongs to $\overline{A^*}(t_i^{-1}\mathcal{R}(M)[t_i^{-1}])$ (see [6, Proposition 1.9]). Thus, $P = \mathcal{P} \cap R$ belongs to $\overline{A^*}(M)$, again by Proposition 3.4.1, which gives what we want. The second statement in the theorem follows as a special case. □

3.5. Asymptotic primes and analytic spread

In this subsection we want to give a version for M of McAdam's theorem concerning membership in $\overline{A^*}(I)$, $I \subseteq R$, an ideal (see [5, Proposition 4.1]). When R is a locally quasi-unmixed domain, then in [11], Rees showed that for a prime P in the support of F/M, P is the center of an essential valuation of $\mathcal{R}(M)$ if and only if the expected local condition on analytic spread holds, i.e., $l(M_P) = \text{height}(P) + r - 1$. Thus, in this case, one gets McAdam's theorem for M by applying Theorem 3.3.3. The general case for M will follow by reducing to this case using various results from section three.

Theorem 3.5.1. *Let R be a Noetherian ring and M be a submodule of $F = R^r$. Assume* $\text{height}(I_r(M)) > 0$ *and let P be a prime ideal in R that contains $I_r(M)$. If $l(M_P) = \text{height}(P) + r - 1$, then $P \in \overline{A^*}(M)$. Conversely, if R is locally quasi-unmixed and $P \in \text{Ass}(F_n/\overline{M_n})$ for some n, then $l(M_P) = \text{height}(P) + r - 1$.*

Proof. We may localize R at P to assume that (R, P) is local at P. We may also assume that the residue field R/P is infinite. Let N be a minimal reduction of M. Then $\overline{N_n} = \overline{M_n}$ for all n, so $\overline{A^*}(N) = \overline{A^*}(M)$ and $l(N) = l(M)$. Thus, it is enough to show the result when $M = N$. Since R/P is infinite, Proposition 2.4.3 gives $\mu(M) = l(M)$, M is generated by analytically independent elements and $P\mathcal{R}$ is a prime ideal.

Assume $l(M) = \text{height}(P) + r - 1$. By definition, $l(M) = \dim(\mathcal{R}/P\mathcal{R})$. Therefore,

$$\text{height}(P\mathcal{R}) \leqslant \dim(\mathcal{R}) - l(M) = d + r - (d + r - 1) = 1.$$

Since $\text{height}(P) > 0$, we have $\text{height}(P\mathcal{R}) = 1$. Let $\mathcal{Q} \subseteq P\mathcal{R}$ be minimal prime. It follows that there exists a height one prime $\overline{\mathcal{P}} \subseteq \overline{\mathcal{R}/\mathcal{Q}}$ with $\overline{\mathcal{P}} \cap \mathcal{R}/\mathcal{Q} = P\mathcal{R}/\mathcal{Q}$. Thus, we have an essential valuation $\mathcal{V}$ of $\mathcal{R}/\mathcal{Q}$, centered on P, such that $\mathcal{V} \cap K$ is a Rees valuation of M, where K is the quotient field of $R/(\mathcal{Q} \cap R)$. Therefore, by Theorem 3.3.6, $P \in \overline{A^*}(M)$.

Now assume that R is quasi-unmixed and $P \in \overline{A^*}(M)$. By Proposition 3.3.5, there exists a minimal prime ideal q such that if we write $S := R/q$, $PS \in \overline{A^*}(MS)$. By Theorem 3.3.3, PS is the center of a Rees valuation of MS, which by definition, means that PS is also the center of an essential valuation $\mathcal{V}$ of $\mathcal{R}(MS)$ for which $M\mathcal{V} \neq F\mathcal{V}$. By [11, Theorem 2.4], $l(MS) = \text{height}(PS) + r - 1$. Since R is quasi-unmixed, P and PS have the same height. Thus by Proposition 2.4.6, we have

$$\text{height}(P) = \text{height}(PS) = l(MS) - r + 1 \leqslant l(M) - r + 1.$$

Since $l(M) \leqslant \text{height}(P) + r - 1$ (Proposition 2.4.5), this gives the result. □

We now summarize the characterizations of $\overline{A^*}(M)$ that we have obtained.

Theorem 3.5.2. *Let R be a Noetherian ring and let M be a submodule of $F = R^r$. Assume* $\operatorname{height}(I_r(M)) > 0$ *and let $P \subseteq R$ be a prime ideal. Then the following are equivalent*:

(i) $P \in \overline{A^*}(M)$.
(ii) *P is the center of a Rees valuation of M.*
(iii) *P contains $I_r(M)$ and $P = \mathcal{P} \cap R$, for some $\mathcal{P} \in \overline{A^*}(x\mathcal{R})$, some $x \in R$ such that* $\operatorname{height}(xR) > 0$.
(iv) *$P/q \in \overline{A^*}((M + qF)/qF)$, for some minimal prime $q \subseteq R$.*
(v) *There exists $1 \leqslant i \leqslant r$ and a prime ideal $\mathcal{P} \in \overline{A^*}(t_i^{-1}\mathcal{R}[t_i^{-1}])$ such that $\mathcal{P} \cap R = P$.*
(vi) *There exists a faithfully flat extension T of R and a prime $Q \in \overline{A^*}(MT)$ with $P = Q \cap R$.*

Furthermore, $I_r(M) \subseteq P$ and $\operatorname{height}(P) = l(M_P) - r + 1$ *imply* (i) *and if R is locally quasi-unmixed, the converse holds.*

3.6. Two applications

In this subsection we will utilize our characterizations of $\overline{A^*}(M)$ derived in the previous subsections to prove that $\overline{A^*}(M)$ is a subset of each of the sets $A^*(M)$ and $\overline{A^*}(I_r(M))$. The proof that $\overline{A^*}(M) \subseteq \overline{A^*}(I_r(M))$ will be accomplished by using the fact that when R is a normal Noetherian domain, the Rees valuations of M are a subset of the Rees valuations of $I_r(M)$ (see [7]).

We begin by showing $\overline{A^*}(M) \subseteq A^*(M)$, thereby extending an important result of Ratliff from the case of ideals (see [10]) to modules. Our task would be made much easier of we knew that the following statement, similar in spirit to Proposition 3.4.1, were true. For a prime $P \subseteq R$, $P \in A^*(M)$ if and only if for some $1 \leqslant i \leqslant r$, there exists a relevant prime divisor $\mathcal{P}$ of $t_i^{-1}\mathcal{R}[t_i^{-1}]$ such that $\mathcal{P} \cap R = P$. This would correspond exactly to a known characterization of $A^*(I)$ for ideals (see [5]). Unfortunately, we have not been able to prove such a statement. However, the following crucial criterion, similar to Proposition 3.3.1, will ensure that a prime ideal is in $A^*(M)$.

Proposition 3.6.1. *Let R be a Noetherian ring, M a submodule of $F = R^r$ and $P \in \operatorname{Spec}(R)$. Assume there exists $\mathcal{P} \in \operatorname{Spec}(\mathcal{R})$ satisfying the following conditions*:

(i) $\mathcal{P} \cap R = P$,
(ii) *there exists $f \in \mathcal{R}_1$ with coefficients in P such that $f \notin \mathcal{P}$,*
(iii) *there exists a non-zerodivisor $x \in P$ and a homogeneous element $d \in \mathcal{R}_l$ such that $\mathcal{P} = \sqrt{x\mathcal{R} :_{\mathcal{R}} d}$.*

Then $P \in A^*(M)$.

Proof. First localize R at P to assume (R, P) is local. Now choose n such that $\mathcal{P}^k \subseteq (x\mathcal{R} :_{\mathcal{R}} d)$ for all $k \geqslant n$. In particular, since $P \subseteq \mathcal{P}$ we have $P^k d \subseteq x\mathcal{R} \subseteq x\mathcal{F}$. Since all coefficients of f^k are in P^k, we have that $f^k d$ is divisible by x in $\mathcal{F}$. Say $f^k d = xq$ for some $q \in F_{k+l}$. Assume that $\mathcal{P}$ satisfies the conditions in the statement. Then, $\mathcal{P} = \sqrt{x\mathcal{R} :_{\mathcal{R}} f^k d}$ as $f \notin \mathcal{P}$. Now

$$\left(x\mathcal{R} :_{\mathcal{R}} f^k d\right) = (x\mathcal{R} :_{\mathcal{R}} xq) = (\mathcal{R} :_{\mathcal{R}} q)$$

as x is a non-zero divisor. Note also that $q \notin \mathcal{R}$ as $\mathcal{P}$ is a proper ideal, so that $q \notin M_{k+l}$. Choose m such that $\mathcal{P}^m \subseteq (x\mathcal{R} :_{\mathcal{R}} f^k d)$. Then $P^m \subseteq \mathcal{P}^m \subseteq (\mathcal{R} : q)$ and $q \in F_{k+l} \backslash M_{k+l}$. Since the elements of P are homogeneous of degree zero when considered as elements of $\mathcal{R}$ and q is homogeneous of degree $l+k$, this means that $P^m q \subseteq \mathcal{R} \cap F_{k+l} = M_{k+l}$. Since P is maximal and $q \in F_{k+l} \backslash M_{k+l}$, we have $P \in \mathrm{Ass}(F_{k+l}/M_{k+l})$. Since this is true for all $k \geqslant n$, we have that $P \in A^*(M)$. □

Theorem 3.6.2. *Let R be a Noetherian ring and M be a submodule of $F = R^r$ satisfying* $\mathrm{height}(I_r(M)) > 0$. *Then* $\overline{A^*}(M) \subseteq A^*(M)$.

Proof. Let $P \in \overline{A^*}(M)$ and localize to assume that (R, P) is local. Note that $I_r(M) \subseteq P$, so $\dim(R) > 0$. If $P \in \mathrm{Ass}(R)$, then $P \in A^*(M)$, by Proposition 2.5.1. Thus, we may assume that R has positive depth.

Now, by the definition of Rees valuation and Theorem 3.3.6, there exists a minimal prime $q \subseteq R$ such that if we write $S := R/q$, PS is the center of an essential valuation $\mathcal{V}$ of $\mathcal{R}(MS)$ for which $I_r(MS)\mathcal{V} \neq \mathcal{V}$. Let $\mathcal{P}_S := m_{\mathcal{V}} \cap \mathcal{R}(MS)$ and $\mathcal{P} := m_{\mathcal{V}} \cap \mathcal{R}$. By Proposition 3.1.7, there exists $\tilde{f} \in \mathcal{R}(MS)_1$ with coefficients in PS, yet $\tilde{f} \notin \mathcal{P}_S$. It follows that there exists f, a preimage of $\tilde{f}$, such that $f \in \mathcal{R}_1$, $f \notin \mathcal{P}$ and f has coefficients in P.

On the other hand, let $y \in R$ be a non-zerodivisor. By Theorem 3.3.6, there exists $\mathcal{P} \in \overline{A^*}(y\mathcal{R})$ with $\mathcal{P} \cap R = P$. By Proposition 3.2.6, for some $n \geqslant 1$ and $x := y^n$, $\mathcal{P} = \sqrt{(x\mathcal{R} : d)}$, for some d, which can be taken to be homogeneous, say of degree l. Thus, $P \in A^*(M)$, by Proposition 3.6.1. □

Remark 3.6.3. Let $I \subseteq R$ be an ideal. Then Ratliff's theorem guarantees $\overline{A^*}(I)$ is contained in $A^*(I)$ when $\mathrm{height}(I) > 0$ (see [10, Corollary 2.6]). Our hypothesis in Theorem 3.6.2 that $\mathrm{height}(I_r(M)) > 0$ is the module analogue of this condition.

We now want to show that $\overline{A^*}(M) \subseteq \overline{A^*}(I_r(M))$. The main point is that if R is a normal Noetherian domain, then the Rees valuations of M are a subset of the Rees valuations of $I_r(M)$ (see [7, Theorem 3.4]).

Theorem 3.6.4. *Let R be a Noetherian ring and M be a submodule of $F = R^r$ satisfying* $\mathrm{height}(I_r(M)) > 0$. *Then* $\overline{A^*}(M) \subseteq \overline{A^*}(I_r(M))$.

Proof. Let $P \in \overline{A^*}(M)$. We may assume that R is local at P. Set $I := I_r(M)$. By Theorem 3.4.2 and Proposition 3.3.5, we can find a minimal prime q contained in the completion $\hat{R}$ of R such that for $S := \hat{R}/q$, $PS \in \overline{A^*}(MS)$. By the ideal case (see [5, Proposition 3.18]), if $PS \in \overline{A^*}(IS)$, then $P \in \overline{A^*}(I)$. Thus, changing notation, we may assume that R is a complete local domain. By Theorem 3.3.3, P is the center of a Rees valuation V of M. From the definition of Rees valuation, it is clear that V is also a Rees valuation of $M\overline{R}$. Since $\overline{R}$ is a normal Noetherian domain, V is a Rees valuation of $I\overline{R}$, by [7]. Thus, V is also a Rees valuation of I. Therefore, by the ideal case [5, Proposition 3.20], $P \in \overline{A^*}(I)$, which is what we wanted to prove. □

Corollary 3.6.5. *Let R be a locally quasi-unmixed ring, let M be a rank r submodule of $F = R^r$, and let $P \in \mathrm{Spec}(R)$. If $l(M_P) = \mathrm{height}(P) + r - 1$, then $l((I_r(M))_P) = \mathrm{height}(P)$.*

Proof. Assume that $l(M_P) = \text{height}(P) + r - 1$. Then $P \in \overline{A^*}(M)$ by Theorem 3.5.1. Thus $P \in \overline{A^*}(I_r(M))$ by Theorem 3.6.4. Therefore $l((I_r(M))_P) = \text{height}(P)$ by Proposition 4.1 of [5]. □

4. Asymptotic primes in low dimension

In this section we study $\overline{A^*}(M)$ in two dimensional Cohen–Macaulay local rings and three dimensional regular local rings. In order to do this some generalizations of results due to Sally in [12] are needed, which extend bounds on the number of generators of ideals in Cohen–Macaulay rings to bounds on the number of generators of M.

4.1. Bounds on the number of generators

Let (R, m) be a Noetherian local ring and M be a submodule of $F = R^r$. In the case that $\lambda(F/M) < \infty$, define the *nilpotency degree* of F/M to be the integer t such that $m^t F \subseteq M$ but $m^{t-1} F \nsubseteq M$. If $I \subseteq R$ is an ideal, then the *order* of I, $\text{ord}_R(I)$, is t if $I \subseteq m^t$ but $I \nsubseteq m^{t+1}$.

Let N be a finitely generated R-module, and $I \subseteq R$ be an ideal. If $\lambda(N/IN) < \infty$, then $\lambda(N/I^n N) < \infty$ for all $n \geqslant 1$ and there exists a polynomial $P(n)$ with rational coefficients, whose degree is equal to $\dim(N)$, such that $P(n) = \lambda(N/I^n N)$ for all $n \gg 0$. The *multiplicity* of I on N, denoted $e_N(I)$, is the product of $(\dim(N))!$ and the leading coefficient of P. Recall that $a \in I^t$ is *superficial of degree t* for I with respect to N if there is an integer $c > 0$ such that $(I^n N :_N a) \cap I^c N = I^{n-t} N$ for all $n > c$. It is straightforward to show that if $x \in I$ is superficial of degree one for I with respect to N, then x^t is superficial of degree t for I with respect to N.

Remark 4.1.1. Recall that superficial elements of degree one preserve multiplicity. In fact, let (R, m) be a local Noetherian ring and N a finitely generated R-module with $\dim(N) = d > 1$. Let I be an ideal of R satisfying $\lambda(N/IN) < \infty$ and assume $a \in I^t$ is superficial of degree t for I with respect to N and is chosen so that $\dim(N/aN) = d - 1$. Then $e_N(I) = t \cdot e_{N/aN}(I)$. See [14, Section VIII.8, Lemma 4].

We next give a bound on the minimal number of generators M in terms of the nilpotency degree of F/M and the multiplicity of the ring. This is an analogue of Theorem 1.2 of [12]. Note that the right-hand side of the estimate now requires a factor of r to reflect that fact the rank of M is greater than one.

Lemma 4.1.2. *Let (R, m) be a Cohen–Macaulay local ring of dimension $d > 0$. Let M be a submodule of $F = R^r$ such that $\lambda(F/M) < \infty$, and let t be the nilpotency degree of F/M. Then*

$$\mu(M) \leqslant r\big(t^{d-1} e_R(m) + d - 1\big).$$

Proof. The proof is by induction on d. Without loss of generality we may assume that R/m is infinite. Note that $m \notin \text{Ass}(R)$ as R is Cohen–Macaulay and $d > 0$. Assume $d = 1$. In this case, there exists an $x \in m$ so that xR is a minimal reduction of m and x is a non-zerodivisor. Then $e_R(m) = \lambda(R/xR)$. Thus,

$$\begin{aligned} r \cdot e_R(m) &= \lambda(F/xF) = \lambda(F/xF) + \lambda(xF/xM) - \lambda(F/M) \\ &= \lambda(F/xM) - \lambda(F/M) = \lambda(M/xM). \end{aligned}$$

The exact sequence

$$0 \to mM/xM \to M/xM \to M/mM \to 0$$

gives

$$\mu(M) := \lambda(M/mM) = \lambda(M/xM) - \lambda(mM/xM) = r \cdot e_R(m) - \lambda(mM/xM).$$

Hence, $\mu(M) \leqslant r \cdot e_R(m)$.

Now assume $d > 1$. We may choose x so that x is a non-zerodivisor on R and also a superficial element of degree one for m with respect to R. Pass to the $d-1$ dimensional Cohen–Macaulay local ring R/x^tR. Note that $x^tF \subseteq M$ by the definition of the nilpotency degree t, and $M/x^tF \subseteq F/x^tF$ with F/x^tF a free R/x^tR-module of rank r. Furthermore $\lambda(\frac{F/x^tF}{M/x^tF}) = \lambda(F/M) < \infty$ and by Nakayama's lemma the nilpotency degree of $\frac{F/x^tF}{M/x^tF}$ is t. Hence, by induction

$$\mu\big(M/x^tF\big) \leqslant r\big(t^{d-2}e_{R/x^tR}\big(m/x^tR\big) + d - 2\big).$$

Next observe that $e_{R/x^tR}(m/x^tR) = te_R(m)$ by Remark 4.1.1. Finally, note that $\mu(x^tF) = \operatorname{rank}(F) = r$ and hence

$$\mu(M) \leqslant \mu\big(M/x^tF\big) + \mu\big(x^tF\big) = \mu\big(M/x^tF\big) + r.$$

Therefore

$$\mu(M) \leqslant \mu\big(M/x^tF\big) + r \leqslant r\big(t^{d-2}te_R(m) + d - 2\big) + r = r\big(t^{d-1}e_R(m) + d - 1\big). \qquad \square$$

Using this lemma, a bound on the number of generators of M can be obtained if the quotient, F/M, is Cohen–Macaulay with an annihilator of positive height. This generalizes Theorem 2.1 of [12]. Again, we see the presence of terms involving r that are not in the original expressions.

Proposition 4.1.3. *Let (R, m) be a Cohen–Macaulay local ring of dimension $d > 0$. Let $M \subseteq F = R^r$ be such that F/M is a Cohen–Macaulay R-module and assume* $\operatorname{height}(I_r(M)) > 0$. *Set* $h := \operatorname{height}(I_r(M)) = \operatorname{height}(\operatorname{ann}(F/M)) > 0$. *Then*

$$\mu(M) \leqslant r\big(e_{F/M}(m)^{h-1}e_R(m) + h - 1\big).$$

Proof. Without loss of generality, we may assume R/m is infinite. The proof is by induction on $s = \dim(F/M)$. If $s = 0$ then $\lambda(F/M) < \infty$ and $h = d$. Now let t be the nilpotency degree of F/M. Note that $m^{t-i}F + M \subsetneq m^{t-i-1}F + M$ for $i = 0, \ldots, t-1$. For if $m^{t-i}F + M = m^{t-i-1}F + M$ then we would have

$$m^{t-i-1}F \subseteq m^{t-i}F + M = m \cdot m^{t-i-1}F + M.$$

Nakayama's lemma would then give that $m^{t-i-1}F \subseteq M$, contradicting that t is the nilpotency degree of F/M. Thus we have a strictly increasing chain of length t

$$0 \subsetneq \frac{m^{t-1}F + M}{M} \subsetneq \frac{m^{t-2}F + M}{M} \subsetneq \cdots \subsetneq \frac{mF + M}{M} \subsetneq \frac{F}{M}.$$

Hence $\lambda(F/M) \geqslant t$. The proposition follows in this case by Lemma 4.1.2, since $e_{F/M}(m) = \lambda(F/M) \geqslant t$ and $h = d$.

Now assume $s > 0$. Note then that $\dim(R) > 1$. Take $x \in m$ such that x is a non-zerodivisor on R and F/M, and also superficial for m with respect to R and F/M. We pass to the $d-1$ dimensional Cohen–Macaulay ring R/xR. Note that $(M+xF)/xF \subseteq F/xF$ and $\mu((M+xF)/xF) = \mu(M)$ since x is a non-zerodivisor on F/M. Also note that $\operatorname{ann}(F/M) + xR$ and $\operatorname{ann}(F/(M+xF))$ are equal up to radical. Thus $h = \operatorname{height}_{R/xR}(\operatorname{ann}(F/(M+xF)))$. Now $F/(M+xF)$ is a $s-1$ dimensional Cohen–Macaulay module over R/xR, and hence by induction we have

$$\begin{aligned}\mu(M) = \mu\left(\frac{M+xF}{xF}\right) &\leqslant r\big(e_{F/(M+xF)}(m)^{h-1} e_{R/xR}(m) + h - 1\big)\\ &= r\big(e_{F/M}(m)^{h-1} e_R(m) + h - 1\big).\end{aligned}$$

The last equality follows from our choice of x together with Remark 4.1.1. □

4.2. Stabilizing points for asymptotic primes

Using the bounds from the previous section we are able to find a specific point by which the sets $\operatorname{Ass}(F_n/M_n)$ and $\operatorname{Ass}(F_n/\overline{M_n})$ must have stabilized if the ring is a two dimensional Cohen–Macaulay local ring or a three dimensional regular local ring. First we will need the following lemma, which is a generalization of Lemma 2.14 in [2]. It will allow us to extend a minimal generating set for the nth torsion-free symmetric power of a reduction of M to one for M_n or $\overline{M_n}$.

Lemma 4.2.1. *Let (R, m) be a local Noetherian ring with R/m infinite and let $M \subseteq F = R^r$ with* $\operatorname{rank}(M) = r$. *Assume that $N \subseteq M$ is a minimal reduction of M. Then $N_n \cap mM_n = N_n \cap m\overline{M_n} = mN_n$ for all n.*

Proof. First note that $mN_n \subseteq N_n \cap mM_n \subseteq N_n \cap m\overline{M_n}$ and so it is enough to show that $mN_n = N_n \cap m\overline{M_n}$. Now consider $T = \overline{\mathcal{R}(M)} \cap \mathcal{F} = \bigoplus_{i=0}^{\infty} \overline{M_n}$ and $S = \mathcal{R}(N) = \bigoplus_{i=0}^{\infty} N_n$. Then T is integral over S, and S/mS is a domain, since N is generated by analytically independent elements and mS is prime by Proposition 2.4.3. By lying over there is a prime Q of T such that $Q \cap S = mS$. In particular $mS \subseteq mT \cap S \subseteq Q \cap S = mS$, so $mS = mT \cap S$. Hence $m\overline{M_n} \cap N_n = mN_n$. □

The following is a generalization of Lemma 4.8 in [5] and the proposition following Lemma 2.14 in [2].

Proposition 4.2.2. *Let (R, m) be a two dimensional Cohen–Macaulay ring and $M \subseteq F = R^r$, with* $\operatorname{rank}(M) = r$. *If $m \in \overline{A^*}(M)$, then for all $n \geqslant (e_R(m) - 1)r + 1$, $m \in \operatorname{Ass}(F_n/M_n)$ and $m \in \operatorname{Ass}(F_n/\overline{M_n})$.*

Proof. We may assume that R/m is infinite. Assume that $m \in \overline{A^*}(M)$. Let L be either M_n or $\overline{M_n}$ for some fixed n, and suppose that $m \notin \mathrm{Ass}(F_n/L)$. Then F_n/L is a one dimensional Cohen–Macaulay module. Clearly we must have $\mathrm{height}(\mathrm{ann}(F_n/L)) = 1$. Thus Proposition 4.1.3 yields

$$\mu(L) \leqslant \mathrm{rank}(F_n)e_R(m) = \binom{n+r-1}{r-1}e_R(m).$$

On the other hand, $m \in \overline{A^*}(M)$ implies that $l(M) = 2 + r - 1 = r + 1$ by Theorem 3.5.1, as R is Cohen–Macaulay and hence quasi-unmixed. Let N be a minimal reduction of M. Since R/m is infinite, N is minimally generated by $r + 1$ analytically independent elements. By Lemma 4.2.1 there is an embedding of N_n/mN_n into L/mL and hence $\binom{n+r}{r} = \mu(N_n) \leqslant \mu(L) \leqslant \binom{n+r-1}{r-1}e_R(m)$. Thus $n \leqslant (e_R(m) - 1)r$. Therefore if $n \geqslant (e_R(m) - 1)r + 1$, then $m \in \mathrm{Ass}(F_n/L)$. □

Along the lines of Theorem 2.15 of [2] we obtain the following proposition.

Proposition 4.2.3. *Let (R, m) be a three dimensional regular local ring, and $M \subseteq F = R^r$ with* $\mathrm{rank}(M) = r$ *and* $\mathrm{height}(\mathrm{ann}(F/M)) = 2$. *Suppose that $l(M) = r + 2$ and set $t = \mathrm{ord}_R(\mathrm{ann}(F/M))$. If $F_n/\overline{M_n}$ or F_n/M_n is Cohen–Macaulay then $\frac{n+2r+1}{(r+1)r} \leqslant t$. In particular, if $n > t(r + 1)r - 2r - 1$, then $m \in \mathrm{Ass}(F_n/\overline{M_n})$ and $m \in \mathrm{Ass}(F_n/M_n)$.*

Proof. We prove the statement for $\overline{M_n}$. The proof for M_n is essentially the same. First we may assume that R/m is infinite. Now let N be a minimal reduction of M. Since $l(M) = r + 2$, by Lemma 4.2.1 we have $\mu(\overline{M_n}) \geqslant \mu(N_n) = \binom{n+r+1}{r+1}$.

Set $e := \mathrm{ord}_R(\mathrm{ann}(F_n/\overline{M_n}))$. By choosing $h \in m\backslash m^2$ sufficiently general (e.g., the leading form of h in $\mathcal{R}(m)/m\mathcal{R}(m)$ does not divide the leading form of some element of order e in $\mathrm{ann}(F_n/\overline{M_n})$), we may assume that

$$\mathrm{ord}_{R/hR}\left(\frac{\mathrm{ann}(F_n/\overline{M_n}) + hR}{hR}\right) = e$$

and h is a non-zerodivisor on $F_n/\overline{M_n}$. Let $S = R/hR$, $G = F_n/hF_n$, and set $K = \frac{\overline{M_n}+hF_n}{hF_n}$. Then S is a two dimensional regular local ring and $\mu(K) = \mu(\overline{M_n})$, since h is not a zerodivisor on $F_n/\overline{M_m}$. Next, we have

$$\frac{\mathrm{ann}(F_n/\overline{M_n}) + hR}{hR} \subseteq \mathrm{ann}_S(G/K),$$

so

$$\mathrm{ord}_S\big(\mathrm{ann}_S(G/K)\big) \leqslant \mathrm{ord}_S\left(\frac{\mathrm{ann}(F_n/\overline{M_n}) + hR}{hR}\right) = e.$$

Furthermore $(\mathrm{ann}(F/M))^n \subseteq \mathrm{ann}(F_n/M_n) \subseteq \mathrm{ann}(F_n/\overline{M_n})$ and hence $e \leqslant nt$. Let $g \in \mathrm{ann}_S(G/K)$ such that $g \in m^c\backslash m^{c+1}$ where $c = \mathrm{ord}_S(\mathrm{ann}_S(G/K))$. Then S/gS is a one dimensional Cohen–Macaulay ring with $e_{S/gS}(m) = c$. Noting that G/K is a finite length S/gS module, by Lemma 4.1.2 we get

$$\mu(K/gG) \leqslant \operatorname{rank}(G/gG)c \leqslant \operatorname{rank}(G/gG)e \leqslant \binom{n+r-1}{r-1}nt.$$

Thus

$$\mu(K) \leqslant \mu(K/gG) + \operatorname{rank}(G) \leqslant \binom{n+r-1}{r-1}nt + \binom{n+r-1}{r-1}.$$

Therefore, we have

$$\binom{n+r+1}{r+1} \leqslant \mu(\overline{M_n}) = \mu(K) \leqslant (nt+1)\binom{n+r-1}{r-1}.$$

Simplifying this inequality gives $\frac{n+2r+1}{(r+1)r} \leqslant t$. □

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