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ON THE POLE ASSIGNABILITY PROPERTY OVER COMMUTATIVE RINGS

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This paper is concerned with the pole assignability property in commutative rings. Specifically, a commutative ring R has the pole assignability property iff given an n-dimensional reachable system (F, G) over R and ring elements $r_1, \dots, r_n \in R$, there exists a matrix K such that the characteristic polynomial of the matrix $F+GK$ is $(X-r_1)\cdots(X-r_n)$. The principal theorem of this paper is *Theorem* 3: Let R be a commutative ring with the property that all rank one projective R-modules are free. Then R has the pole assignability property iff given a reachable system (F, G) there is a unimodular vector in the image of G.

In this note we prove three theorems, the last of which is a characterization of the pole assignability property within the class of commutative rings over which all rank one projective modules are free. The proofs of these results depend on ideas scattered throughout several papers (most notably [2–4, 6]). We do feel, however, that the theorems are important enough to warrant a self-contained treatment and we shall endeavor to provide such here. We begin by establishing some terminology.

Given a pair of matrices F and G of sizes $n \times n$ and $n \times m$ (respectively) over a commutative ring R, the n-dimensional system (F, G) is said to be *reachable* if the columns of the matrix $[G, FG, \dots, F^{n-1}G]$ span R^n. The system is *pole-assignable* if for each choice of $r_1, \dots, r_n \in R$, there exists an $m \times n$ matrix K over R such that the matrix $F+GK$ has $(X-r_1)\cdots(X-r_n)$ as its characteristic polynomial. The ring R is said to have the *PA-property* if every reachable system over R is pole-assignable. An $n \times m$ matrix G over R is said to be *good* if there exists an $n \times n$ matrix F such that (F, G) is reachable. We say that R has the *GCU-property* if every good matrix has a unimodular vector in its image. The rest of our terminology will be standard terminology from the theory of linear algebra over commutative rings and we cite McDonald's excellent text [7] as a reference.

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Theorem 1. *If R has the GCU-property, then R has the PA-property.*

Proof. Our proof will involve a succession of results, the first of which was obtained independently by the present authors and the authors of [6] (see [6, Theorem B]).

Lemma 1. *If R has the GCU-property, then each homomorphic image S of R has the property that stably free S-modules are free.*

Proof. By Theorem 1 of [2], any homomorphic image of R has the GCU-property. Thus, it suffices to prove that stably free R-modules are free. So, let P be a stably free R-module. By Gabel's Theorem [7, Theorem IV.44], there exist positive integers m and n such that P^m is isomorphic to R^n as R-modules. We proceed to find a reachable system (F, G) such that the column module of G is isomorphic to P.

Now, without loss of generality, we may assume that $R^n = P \oplus \cdots \oplus P$, m times. Let $g: R^n \to R^n$ be projection onto the first P-factor and let $f: R^n \to R^n$ be defined as $f(p_1, \ldots, p_m) = (p_m, p_1, \ldots, p_{m-1})$. Using $\mathrm{im}(g)$ to denote the image of g, we clearly have

$$\langle \mathrm{im}(g), f(\mathrm{im}(g)), \ldots, f^{n-1}(\mathrm{im}(g)) \rangle = R^n.$$

Thus if F and G are any matrices representing f and g respectively, it follows that (F, G) is a reachable system with the column module of G isomorphic to P. By the GCU-property, the column module of G contains a unimodular vector and consequently P contains a free summand of rank one. By induction on the rank of P, it follows that P is free. □

A definition is in order before continuing to our next result. Let (F, G) be a system over R. Call a system $(\tilde{F}, \tilde{G})$ *systems equivalent* to (F, G) if it is obtained from (F, G) by one of the following three transformations:

(i) $F \mapsto \tilde{F} = AFA^{-1}$, $G \mapsto \tilde{G} = AG$, for invertible A;

(ii) $F \mapsto \tilde{F} = F + GK$, $G \mapsto \tilde{G} = G$, for any K of suitable size;

(iii) $F \mapsto \tilde{F} = F$, $G \mapsto \tilde{G} = GB$, for invertible B.

It is clear that systems equivalence is an equivalence relation and that given any two equivalent systems the first is reachable (resp. pole-assignable) if and only if the second is.

Lemma 2 (cf. [8]). *The ring R has the GCU-property if and only if each reachable system over R is systems equivalent to one of the form*

$$\left(\begin{bmatrix} & & \\ & * & \\ & & \\ \hline 0 & 0 \cdots & 0 \end{bmatrix}, \begin{bmatrix} & & & 0 \\ & & & 0 \\ & * & & \vdots \\ & & & 0 \\ \hline 0 & 0 \cdots & 0 & 1 \end{bmatrix} \right).$$

Proof. The proof of the 'if' implication is easy. Conversely, suppose that (F, G) is a reachable system over R. By the GCU-property, there exist unimodular vectors $u \in R^m$ and $v \in R^n$ such that $Gu = v$. By Lemma 1, stably free R-modules are free and hence unimodular vectors can be extended to bases [7, Theorem IV.41]. In particular, there exist $u_1, \ldots, u_{m-1} \in R^m$ and $v_1, \ldots, v_{n-1} \in R^n$ such that $\{u_1, \ldots, u_{m-1}, u\}$ is a basis for R^m and $\{v_1, \ldots, v_{n-1}, v\}$ is a basis for R^n. Let $U = [u_1, \ldots, u_{m-1}, u]$ and $V = [v_1, \ldots, v_{n-1}, v]$. Then (F, G) is systems equivalent to $(V^{-1}FV, V^{-1}GU)$, where $V^{-1}GU$ has the form

$$\left[\begin{array}{c|c} & 0 \\ * & \vdots \\ & 0 \\ & 1 \end{array}\right].$$

Right multiplication by an invertible matrix shows that this latter system is equivalent to

$$(\tilde{F}, \tilde{G}) = \left(V^{-1}FV, \left[\begin{array}{c|c} & 0 \\ * & \vdots \\ & 0 \\ \hline 0 \cdots 0 & 1 \end{array}\right] \right).$$

Set K equal to the $m \times n$ matrix

$$\left[\begin{array}{c} 0 \\ \hline -a_1, \ldots, -a_n \end{array}\right]$$

where $[a_1, \ldots, a_n]$ is the last row of $\tilde{F}$. Then the system $(\tilde{F} + \tilde{G}K, \tilde{G})$ has the desired form. $\square$

Using a trick from [4], we can now complete the proof of Theorem 1. We use induction on n, the dimension of the reachable system (F, G). If $n = 1$, then (F, G) has the form $([a], [b_1, \ldots, b_m])$ for some $a, b_1, \ldots, b_m \in R$. Since (F, G) is reachable, there exist $t_1, \ldots, t_m \in R$ such that $t_1 b_1 + \cdots + t_m b_m = 1$. If $r \in R$, set

$$K = (r - a) \begin{bmatrix} t_1 \\ \vdots \\ t_m \end{bmatrix}.$$

Then $F + GK = [r]$ and so $\det(XI - (F + GK)) = X - r$.

If $n > 1$ we can, by Lemma 2, assume that (F, G) has the form

$$\left(\left[\begin{array}{c|c} F_1 & G_1 \\ \hline 0 & 0 \end{array}\right], \left[\begin{array}{c|c} & 0 \\ G_2 & \vdots \\ & 0 \\ \hline 0 \cdots 0 & 1 \end{array}\right] \right)$$

where F_1 is $(n-1) \times (n-1)$, G_2 is $(n-1) \times (m-1)$ and $G_1 \in R^{n-1}$.

It is readily seen [4, Lemma 1] that the system $(F_1, [G_1, G_2])$ is reachable. If $r_1, \dots, r_{n-1}, r_n \in R$ are given, there exists, by induction on n, an $m \times (n-1)$ matrix $\tilde{K}$ such that $F_1 + [G_1, G_2]\tilde{K}$ has characteristic polynomial $(X - r_1) \cdots (X - r_{n-1})$. Write

$$\tilde{K} = \left[\begin{array}{c} K_1 \\ \hline K_2 \end{array}\right]$$

where K_1 is $1 \times (n-1)$ and K_2 is $(m-1) \times (n-1)$. Then

$$F_1 + [G_1, G_2]\tilde{K} = F_1 + G_1 K_1 + G_2 K_2.$$

Note that (F, G) is systems equivalent to

$$\left(\left[\begin{array}{c|c} I & 0 \\ \hline -K_1 & 1 \end{array}\right] \left[\begin{array}{c|c} F_1 & G_1 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} I & 0 \\ \hline K_1 & 1 \end{array}\right], \left[\begin{array}{c|c} I & 0 \\ \hline -K_1 & 1 \end{array}\right] \left[\begin{array}{c|c} G_2 & 0 \\ \hline 0 & 1 \end{array}\right] \right)$$
$$= \left(\left[\begin{array}{c|c} F_1 + G_1 K_1 & G_1 \\ \hline * & * \end{array}\right], \left[\begin{array}{c|c} G_2 & 0 \\ \hline -K_1 G_2 & 1 \end{array}\right] \right).$$

The latter system is equivalent to

$$(\tilde{F}, \tilde{G}) = \left(\left[\begin{array}{c|c} F_1 + G_1 K_1 & G_1 \\ \hline * & * \end{array}\right] + \left[\begin{array}{c|c} G_2 & 0 \\ \hline -K_1 G_2 & 1 \end{array}\right] \left[\begin{array}{c|c} K_2 & 0 \\ \hline 0 & 0 \end{array}\right], \left[\begin{array}{c|c} G_2 & 0 \\ \hline -K_1 G_2 & 1 \end{array}\right] \left[\begin{array}{c|c} I & 0 \\ \hline K_1 G_2 & 1 \end{array}\right] \right)$$
$$= \left(\left[\begin{array}{c|c} F_1 + G_1 K_1 + G_2 K_2 & G_1 \\ \hline * & * \end{array}\right], G \right).$$

Let K be the $m \times n$ matrix

$$\left[\begin{array}{c} 0 \\ \hline -a_1, \dots, -a_{n-1}, r_n - a_n \end{array}\right]$$

where $[a_1, \dots, a_n]$ is the last row of $\tilde{F}$. Then

$$\tilde{F} + GK = \left[\begin{array}{c|c} F_1 + G_1 K_1 + G_2 K_2 & G_1 \\ \hline 0 \cdots 0 & r_n \end{array}\right]$$

which has the desired characteristic polynomial $(X - r_1) \cdots (X - r_n)$. $\square$

As one might expect, there are examples of rings having the PA-property, but not the GCU-property. Indeed, any Dedekind domain with non-trivial 2-torsion in its class group is such an example [2].

There is a weakened form of the converse which is valid. Before proving it, we need some additional terminology. If A is a matrix over a ring R, then, following [5], we shall say that A is a $(*)$-*matrix* if the content of A is R - that is, no maximal ideal of R contains all the entries of A - *and* all 2×2 minors of A are zero. The ring R is said to have the *GCS-property* if and only if given a good matrix G over R, there exists a matrix V such that GV is a $(*)$-matrix.

In Theorem 2 below we show that rings with the PA-property have the GCS-property. This should be compared with Corollary 3.6 in [3] and Theorem A in [6].

Theorem 2. *If R has the PA-property, then R has the GCS-property.*

Proof. Let G be an $n\times m$ good matrix over R. We must find a matrix V such that GV is a $(*)$-matrix. Now, since G is good and R has the PA-property, there exist an $n\times n$ matrix F and an $m\times n$ matrix K such that (F,G) is reachable and $F+GK$ has characteristic polynomial $(X-1)^n$. Thus, $F+GK$ is invertible. Since $(F+GK,G)$ is also reachable, we may assume from the start that F is invertible.

We now construct a matrix A as in the proof of Proposition 3.3 in [3]. If I is the $n\times n$ identity matrix, then $(F+I,G)$ is reachable. By the PA-property there is an $m\times n$ matrix K' for which $A=F+I+GK'$ has characteristic polynomial $X^{n-1}(X-1)$. By the Cayley–Hamilton theorem, $A^{n-1}(A-I)=0$ and it follows that A^{n-1} is an idempotent matrix. We claim, moreover, that A^{n-1} is a $(*)$-matrix.

First we show that A^{n-1} has unit content. If not, there exists a maximal ideal M of R containing the entries of A^{n-1}. Using $\bar{}$ to denote reduction modulo M, we have that $\bar{A}^{n-1}=\bar{0}$. Thus the minimal polynomial for $\bar{A}$ over the field $\bar{R}$ has no non-zero roots. On the other hand, the characteristic polynomial for $\bar{A}$ is $X^{n-1}(X-\bar{1})$ which has the non-zero root $\bar{1}$. This is a contradiction, since over a field the roots of the characteristic polynomial and minimal polynomial must agree.

Next we show that all 2×2 minors of A^{n-1} are zero. We claim that the ideal J of R generated by the 2×2 minors of A^{n-1} is contained in every prime ideal of R. Let P be a prime ideal of R, L the quotient field of R/P and $\bar{}$ denote reduction modulo P. Since the characteristic roots of $\bar{A}$ are in L, there exists an invertible matrix Q over L with

$$Q\bar{A}Q^{-1}=\left[\begin{array}{cccc|c} 0 & 1 & & 0 & 0\\ & \ddots & \ddots & & \vdots\\ & & \ddots & 1 & \\ 0 & & & 0 & 0\\ \hline 0 & & \cdots & 0 & 1\end{array}\right]$$

the Jordan Canonical Form for $\bar{A}$ over L. Thus

$$Q\bar{A}^{n-1}Q^{-1}=\left[\begin{array}{c|c} & 0\\ 0 & \vdots\\ & 0\\ \hline 0\cdots 0 & 1\end{array}\right]$$

and the 2×2 minors of $Q\bar{A}^{n-1}Q^{-1}$ are zero. Hence the 2×2 minors of $\bar{A}^{n-1}$ vanish (since over the field L this is equivalent to rank $\bar{A}^{n-1}=1$). Thus $J\subseteq P$ as claimed.

Now, since J is finitely generated and is contained in each prime ideal of R, there exists a positive integer N such that $J^N=0$. On the other hand, since A^{n-1} is an idempotent matrix, the Binet–Cauchy formula [7, Theorem 1.5] implies that J is an idempotent ideal. Hence $J=J^2=\cdots=J^N=0$, and all 2×2 minors of A^{n-1} are zero.

Therefore, A^{n-1} is a $(*)$-matrix.

To finish, we have $0=(A-I)A^{n-1}=(F+GK')A^{n-1}$, so $GK'A^{n-1}=-FA^{n-1}$. Since F is invertible, it follows that $-FA^{n-1}$ is also a $(*)$-matrix (use the Binet–Cauchy formula to see that the 2×2 minors of $-FA^{n-1}$ are also zero). Setting $V=K'A^{n-1}$, we have that GV is a $(*)$-matrix and the proof is complete. □

By placing a rather weak hypothesis on the ring R, we are able to obtain our promised characterization of the PA-property.

Theorem 3. *Let R be a commutative ring over which all rank one projective modules are free. Then R has the PA-property if and only if R has the GCU-property.*

Proof. If R has the GCU-property, then R has the PA-property by Theorem 1 (regardless of R having rank one projective R-modules free).

Now suppose that R has rank one projective modules free and the PA-property. Let G be a good matrix over R. Then just as in the proof of Theorem 2, we arrive at $A=F+I+GK'$, where F is invertible and A^{n-1} is an idempotent $(*)$-matrix. Since A^{n-1} is idempotent, its column module C is a projective summand of R^n and since the 2×2 minors of A^{n-1} vanish, C has rank one. Thus, C is free of rank one and so $C=R\cdot \upsilon$ for some $\upsilon\in R^n$. Moreover, since A^{n-1} has unit content, υ must be unimodular. Thus, there is a unimodular vector in the image of A^{n-1}. Finally, since F is invertible and $GK'A^{n-1}=-FA^{n-1}$, it follows that G has a unimodular vector in its image. The proof is complete. □

Remark. One specific consequence of these results is the following:

If a ring R has rank one projective R-modules free and the PA-property, then stably free R-modules are free.

This follows from Theorem 3 and Lemma 1. In particular, if

$$R=\mathbb{R}[X,Y,Z]/(X^2+Y^2+Z^2-1),$$

the coordinate ring of the real 2-sphere, then R does not have the PA-property.

Remark. Theorem 5 of [2] states that if a ring R has 1 in its stable range, then R has the FC-property (a strong form of the PA-property) if and only if R has the GCU-property. This result should be compared with Theorem 3 of the present paper. Since there are no formal implications between the hypotheses "rank one projective R-modules are free" and "R has 1 in its stable range", the relationship between these theorems is unclear.

We close with the following conjecture.

Conjecture. Let R be a commutative ring. Then R has the PA-property if and only if R has the GCS-property.

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