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Pole Assignability in Polynomial Rings, Power Series Rings, and Prüfer Domains

JAMES BREWER*, DANIEL KATZ†, AND WILLIAM ULLERY

University of Kansas, Lawrence, Kansas 66045

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The topics treated in this paper have their origins in the area of algebraic systems theory. However, the paper itself should be classified as pure commutative algebra and we shall present it as such in the body of the text. Still, it is appropriate to give a brief paragraph of motivation.

If a physical system is governed by a pair (F, G) of matrices, then the stability of the system can be determined by examining the eigenvalues of the matrix F. If the system is unstable, a "feedback" matrix K can sometimes be employed in such a way that the eigenvalues of the matrix $F+GK$ measure the stability of the (modified) system $(F+GK, G)$. In this manner, an unstable system can be rendered stable. The pole assignability problem over commutative rings is one method of attacking the problem of finding such matrices K.

The paper is divided into four sections. Section 1 is given over almost entirely to defining the properties in which we shall be interested. It concludes with a theorem about residuating and lifting the properties. Section 2 is concerned with the preservation of the properties under polynomial ring and power series ring formation. Section 3 is concerned with "feedback cyclization," a strong form of pole assignability. Section 4 is concerned with pole assignability over Prüfer domains.

A complete elaboration of our results must await the introduction of the necessary terminology. For now, we mention the following results in somewhat vague language. If R is a zero-dimensional ring, then the pole assignability problem is solvable in $R[X]$ and in $R[[X]]$. From a ring-theoretic standpoint, almost any class of commutative rings contains members for which the feedback cyclization problem is solvable. On the other hand, if R is a ring with 1 in its stable range, then the feedback cyclization problem is solvable in R if and only if a certain nice matricial property

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holds in R. In Prüfer domains, the pole assignability problem can be solved if "simultaneous bases" for projective submodules of free modules can be found. This appears to open up a problem for Prüfer domains similar to the old one for Bézout domains: Is every Bézout domain an elementary divisor domain?

1. Introduction

At this point it is necessary to introduce a large amount of notation and terminology. We proceed to do just that, but some readers might wish to skip to the results themselves, returning to this section when necessary.

Let R be a commutative ring with identity. By a *system* over R we shall mean a pair (F, G) of matrices over R where F is $n \times n$ and G is $n \times m$, n, m positive integers. The system is *reachable* if and only if the R-module generated by the columns of the matrix $[G, FG, ..., F^{n-1}G]$ is R^n. Thus, (F, G) is reachable if and only if the map from R^{mn} to R^n determined by the matrix $[G, FG, ..., F^{n-1}G]$ is surjective. The system (F, G) is called *pole assignable* if and only if given $r_1, ..., r_n \in R$, there exists a "feedback" matrix K over R such that the characteristic polynomial of the matrix $F+GK$ is $(X-r_1)\cdots(X-r_n)$. It is a fact (see Bumby *et al.*, [6]) that if the system (F, G) is pole assignable, it is reachable. In some sense, this paper is interested in determining those rings for which the converse is true. The system (F, G) is *coefficient assignable* if and only if given $r_0, r_1, ..., r_{n-1} \in R$, there exists a feedback matrix K such that the characteristic polynomial of the matrix $F+GK$ is $r_0 + r_1 X + \cdots + r_{n-1} X^{n-1} + X^n$. If (F, G) is a system over R, we say that "we can feedback to a cyclic vector" if there exists a vector $u \in R^m$ and a feedback matrix K such that Gu is a cyclic vector for the matrix $F+GK$—that is, the matrix $[Gu, (F+GK)Gu, ..., (F+GK)^{n-1} Gu]$ has unit determinant.

It is not hard to see that if the system (F, G) feeds back to a cyclic vector, then (F, G) is coefficient assignable (e.g., see Brewer, Bunce, and Van Vleck [1]). It is evident that if (F, G) is coefficient assignable, it is pole assignable.

We say that a ring R has the *PA-property* if each reachable system over R is pole assignable. We say that R has the *CA-property* if each reachable system over R is coefficient assignable. We say that R has the *FC-property* if each reachable system over R feeds back to a cyclic vector. These topics are defined and discussed in the paper [6] and that paper should be on hand when reading any paper in this area. There have been several recently, among them Brewer, Naudé, and Naudé [4], Tannenbaum [21], Brewer, Heinzer, and Lantz [3], Naudé and Naudé [20], and Hautus and Sontag [15].

We need to define only a few more notions. As before, let R be a commutative ring. We shall say that an $n \times m$ matrix G over R is *good* if there exists an $n \times n$ matrix F over R such that the system (F, G) is reachable. It is clear that a good matrix has unit content—that is, the ideal of R generated by the entries of a good matrix is R. It is proved in [4] that if D is a Bézout domain, then D has the PA-property if and only if given a good $n \times m$ matrix G over D, there exists a vector $u \in D^m$ such that Gu is unimodular; that is, the content of Gu is D. We abstract this notion. If R is a commutative ring, then we say that R has the *GCU-property* if and only if whenever G is a good $n \times m$ matrix over R, there exists a vector $u \in R^m$ such that Gu is unimodular.

In [15], it is shown that any Dedekind domain has the PA-property. The proof of this lovely result involves two steps: First showing that a Dedekind domain has what the authors call "property (†)"; second, showing that property (†) implies the PA-property for any ring. Set in our framework, property (†) says the following. If G is a matrix having unit content, then there exists a matrix V such that the matrix GV is a (*)-matrix in the notation of Gilmer and Heitmann [12]—that is, the content of GV is the whole ring and all 2×2 minors of GV are zero. If a ring R has property (†), we shall say that R has the *UCS-property*. (We are doing this for consistency's sake only. The property (†) notation is fine and we do not mean to imply otherwise.)

Finally, we introduce one further bit of terminology. It is merely a strong form of the UCS-property. If a ring is such that given an $n \times m$ matrix G of unit content, there exists a vector $u \in R^m$ such that Gu is unimodular, then we shall say that R has the *UCU-property*. As noted above, if R has the UCU-property, then R has the UCS-property (and hence the PA-property). It is also clear that if R has the UCU-property, then R has the GCU-property.

We conclude this introductory section with a basic theorem which says that these properties respect homomorphic images and that some of them lift modulo the Jacobson radical.

THEOREM 1. *Let R be a commutative ring with I an ideal of R and let J denote the Jacobson radical of R.*

(1) *If R has the PA- (resp. CA-, FC-, GCU-, UCS-, UCU-), property, then R/I has the PA- (resp. CA-, FC-, GCU-, UCS-, UCU-) property.*

(2) *If R/J has the GCU- (resp. UCU-, FC-) property, then R has the GCU- (resp. UCU-, FC-) property.*

Proof. (1) We will prove only that if R has the PA-property, so does R/I, the proofs of the remaining assertions being analogous. Thus, let

$(\bar{F}, \bar{G})$ be a reachable system over R/I and let $\bar{r}_1, \ldots, \bar{r}_n \in R/I$. (Here, of course, $\bar{}$ denotes reduction of elements or matrix entries modulo I.) Since the (R/I)-module generated by the columns of the matrix $[\bar{G}, \bar{F}\bar{G}, \ldots, \bar{F}^{n-1}\bar{G}]$ is $(R/I)^n$, there exist column vectors $C_1, \ldots, C_k$, each of which belongs to IR^n, such that the system $(F, [G, C_1, \ldots, C_k])$ is reachable over R. Because R has the PA-property, there exists an $(m+k) \times n$ matrix $K = [\begin{smallmatrix} K_1 \\ K_2 \end{smallmatrix}]$, K_1 being $m \times n$, K_2 being $k \times n$, such that the characteristic polynomial of $F + [G, C_1, \ldots, C_k] K$ is $(X - r_1) \cdots (X - r_n)$. But then $\bar{F} + \overline{[G, C_1, \ldots, C_k] K} = \bar{F} + [\bar{G}, \bar{0}, \ldots, \bar{0}][\begin{smallmatrix} \bar{K}_1 \\ \bar{K}_2 \end{smallmatrix}] = \bar{F} + \bar{G}\bar{K}_1$ has characteristic polynomial $(X - \bar{r}_1) \cdots (X - \bar{r}_n)$ as desired.

(2) If G is an $n \times m$ good matrix over R, then $\bar{G}$ is good over R/J (where $\bar{}$ denotes reduction of matrix entries modulo J). Hence, there exists a vector $\bar{v} \in (R/J)^m$ such that $\bar{G}\bar{v}$ is unimodular over R/J. Clearly Gv is unimodular over R.

That the UCU and FC-properties lift is just as easy.

2. Polynomial Rings and Power Series Rings

This section actually divides into two subsections, each dealing with one of the topics of the title.

The principal result of the first part is that if R has (Krull) dimension zero, then $R[X]$ has the UCU-property. (We conjecture the converse, but have been unable to prove it.)

In a sense, the origin of this entire area is the proof in Morse [19] that if F is a field, then $F[X]$ has the PA-property. Also, it is shown in [21] that $F[X, Y]$ never has the PA-property, independent of the field F. In fact if R is any commutative ring, let M be a maximal ideal of R. If $R[X, Y]$ had the PA-property, then so would $(R/M)[X, Y]$ by Theorem 1. Consequently, for any R, $R[X, Y]$ never has the PA-property, and in particular, never has the UCU-property. Thus, our Theorem 2, along with its conjectured converse, would complete the story of the UCU-property in polynomial rings.

Recall from Kaplansky [16] that a ring R is called an *elementary divisor ring* if each matrix A over R admits a diagonal reduction—that is, there exist invertible matrices P and Q such that PAQ is diagonal.

THEOREM 2. *Let R be a ring of dimension zero. Then the polynomial ring $R[X]$ has the UCU-property.*

Proof. Let N be the nilradical of R. Since R is zero-dimensional, N is also the Jacobson radical of R. Hence, $N[X]$ is the Jacobson radical of $R[X]$. By Theorem 1, it suffices to prove that $R[X]/N[X] \cong (R/N)[X]$

has the UCU-property. Now, R/N is a von Neumann regular ring and so we may assume that R itself is. We claim that if R is von Neumann regular, then $R[X]$ is an elementary divisor ring. Once this has been justified, the result will follow because of a standard argument showing that elementary divisor rings have the UCU-property. (See [6, p. 119].)

Let A be a matrix over $R[X]$. Set $I = \{a \in R \mid A$ admits a diagonal reduction over $R_a[X]\}$, where R_a denotes the localization of R at the element a, and let I^* be the ideal generated by I. For each maximal ideal M of R, $R_M[X]$ is a PID and so A admits a diagonal reduction over $R_M[X]$. It follows easily from this that $I^* = R$, so there exist elements $a_i \in I$ and $r_i \in R$ with $r_1 a_1 + \cdots + r_n a_n = 1$. Since R is von Neumann regular, we may assume that each a_i is idempotent. Replacing $a_1, a_2, \dots, a_n$ by $a_1, (1-a_1) a_2, \dots, (1-a_1)(1-a_2) \cdots (1-a_{n-1}) a_n$, respectively, we may assume that the a_i's are pairwise orthogonal idempotents. Thus, $R = Ra_1 \oplus \cdots \oplus Ra_n \cong R_{a_1} \oplus \cdots \oplus R_{a_n}$ via a "natural" isomorphism. It follows that A admits a diagonal reduction over $R[X]$. This completes the proof of the theorem.

Remark. We note that if $R[X]$ is an elementary divisor ring, then R is von Neumann regular. In fact, in Gilmer and Parker [13] it is shown that if $R[X]$ is a Prüfer ring, then R is von Neumann regular. Thus, R is a von Neumann regular ring if and only if $R[X]$ is an elementary divisor ring. In the sequel, we shall look at the analogous question for power series rings.

Note that if S is a ring with the UCU-property, then all projective S-modules of constant (finite) rank are free. (See Lemma 8.) In particular, it follows from Theorem 2 that if R is zero-dimensional, then constant rank projective $R[X]$-modules are free. Now, as remarked earlier, $R[X, Y]$ does not have the PA-property, much less the UCU-property. Nevertheless, if R is zero-dimensional, constant rank projective $R[X, Y]$-modules are free as we now prove.

PROPOSITION 1. *Let R be a ring of dimension zero and let $X_1, \dots, X_n$ be indeterminates. Then projective modules of constant rank over $R[X_1, \dots, X_n]$ are free.*

Proof. Let P^* be a projective $R[X_1, \dots, X_n]$-module of constant (finite) rank. Then, as is well known, P^* is finitely generated and hence extended by Brewer and Costa [2, Corollary 2]. Write $P^* \cong P \otimes_R R[X_1, \dots, X_n]$ where P is a projective R-module. Since constant rank projectives over R are free (McDonald and Waterhouse [18]), we need only show that P is of constant rank. First observe that $P \cong P^*/(X_1, \dots, X_n) P^*$, as can be seen via a standard calculation with tensor products. Let M be a maximal ideal of R. Then $(M, X_1, \dots, X_n) = M^*$ is a maximal ideal of $R[X_1, \dots, X_n]$ and

$R/M \cong R[X_1,..., X_n]/M^*$. Moreover, $P/MP \cong P^*/M^*P^*$. Therefore, $\dim_{R/M}(P/MP) = \dim_{R[X_1,...,X_n]/M^*}(P^*/M^*P^*)$. Since P^* has constant rank, it follows that P does also.

We begin our treatment of power series with a definitive result for some of the properties we are interested in.

THEOREM 3. *Let R be a commutative ring and denote by $R[[X]]$ the power series ring in one variable over R. Then,*

(a) *R has the FC-property if and only if $R[[X]]$ has.*

(b) *R has the GCU-property if and only if $R[[X]]$ has.*

(c) *R has the UCU-property if and only if $R[[X]]$ has.*

Proof. If J is the Jacobson radical of R, then $J+(X)$ is the Jacobson radical of $R[[X]]$ and $R/J \cong R[[X]]/(J+(X))$. The result is now immediate from Theorem 1.

Remark. Let R be a zero-dimensional ring. Since R modulo its Jacobson radical is von Neumann regular, R has the *UCU*-property. It follows from Theorem 3 that $R[[X]]$, like $R[X]$, has the *UCU*-property.

It will follow from the results in Section 3 that a zero-dimensional ring R has the *FC*-property. Consequently, Theorem 3 implies that $R[[X]]$ has the *FC*-property. However, $R[X]$ need not have the *FC*-property. For example, if $\mathbb{R}$ is the real field, then $\mathbb{R}[X]$ does not have the *FC*-property (see [1] for a proof).

We conjecture that if R has the *UCS*-property, then so does $R[[X]]$. So far, a proof has eluded us. As for the *PA*-property, we have little feel. However, two ways in which $R[[X]]$ could have the *PA*-property are:

If R is an elementary divisor ring, then R has the *UCU*-property and by Theorem 3, so does $R[[X]]$. Therefore, $R[[X]]$ has the *PA*-property. In particular, if $\mathbb{Z}$ is the ring of integers, then $\mathbb{Z}[[X]]$ has the *PA*-property, fails to have the *FC*-property, and is not an elementary divisor ring. Adjoining additional indeterminates yields such examples of arbitrary dimension.

If $R[[X]]$ is itself an elementary divisor ring, then $R[[X]]$ has the *PA*-property. Indeed, our original interest in power series rings was kindled by a desire to use them to give an example of a Bézout ring that failed to have the *PA*-property. As we shall see, if $R[[X]]$ is a Bézout ring, it must be an elementary divisor ring and so our idea was doomed to failure. At any rate, in the process an interesting theorem emerged. We proceed to its proof by a succession of results.

LEMMA 1. *If $R[[X]]$ is a Bézout ring, then R is von Neumann regular.*

Proof. For every $r \in R$, (r, X) is a principal regular ideal, hence invertible. Therefore, $(r, X)^2 = (r^2, X^2)$. In particular, $rX = fr^2 + gX^2$ for some f, $g \in R[[X]]$. If $f = a_0 + a_1 X + a_2 X^2 + \cdots$, where $a_i \in R$, then $rX = a_0 r^2 + a_1 r^2 + hX^2$ for some $h \in R[[X]]$. Equating corresponding coefficients yields $r = a_1 r^2$, so R is regular.

Call a ring R *Hermite* if every matrix over R can be lower triangulated; i.e., for every matrix A over R, there exists an invertible matrix P such that AP is lower triangular. This is equivalent to the condition that every 1×2 matrix over R can be diagonalized by right multiplication by an invertible matrix [16]. Observe that von Neumann regular rings are Hermite. This follows easily from the fact that every nonzero element of a von Neumann regular ring is (uniquely) the product of an idempotent with a unit. Also, it is clear that Hermite rings are Bézout rings. Our next result shows that the converse is true for power series rings.

PROPOSITION 2. *$R[[X]]$ is an Hermite ring if and only if it is a Bézout ring.*

Proof. As mentioned above, Hermite rings are always Bézout rings. So, we assume that $R[[X]]$ is a Bézout ring and show that it is an Hermite ring. Suppose $[f\ g]$ is a 1×2 matrix over $R[[X]]$ where $f = \sum f_i X^i$ and $g = \sum g_i X^i$. By Lemma 1, R is regular. Hence R is Hermite and there exist $a_0, b_0, c_0, d_0 \in R$ such that $[f_0\ g_0]\left[\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix}\right] = [a_0 f_0 + c_0 g_0\ 0]$ and $a_0 d_0 - b_0 c_0$ is a unit.

Set $R_1 = \prod\{R/M \mid M \text{ is a maximal ideal of } R\}$. Since R is reduced and zero-dimensional, R may be viewed as a subring (and hence as an R-submodule) of R_1. Consider the following countable system of equations with coefficients in R in the unknowns $b_1 b_2, \ldots; d_1, d_2, \ldots$:

$$\begin{aligned} f_1 b_0 + f_0 b_1 + g_1 d_0 + g_0 d_1 &= 0, \\ f_2 b_0 + f_1 b_1 + f_0 b_2 + g_2 d_0 + g_1 d_1 + g_0 d_2 &= 0, \\ f_3 b_0 + f_2 b_1 + f_1 b_2 + f_0 b_3 + g_3 d_0 + g_2 d_1 + g_1 d_2 + g_0 d_3 &= 0, \\ &\vdots \end{aligned} \tag{$*$}$$

Since R_1 is a product of fields, $(*)$ clearly has a solution over R_1. $R[[X]]$ a Bézout ring and R regular imply that R is an $\aleph_0$-pure submodule of R_1 by Theorem 1 of [5]. This means that $(*)$ has a solution over R. Thus, there exist b_i and d_i in R such that $[f\ g]\left[\begin{smallmatrix} a_0 & b \\ c_0 & d \end{smallmatrix}\right] = [a_0 f + c_0 g\ 0]$, where $b = \sum_{i=0}^{\infty} b_i X^i$ and $d = \sum_{i=0}^{\infty} d_i X^i$. Moreover, $a_0 d - b c_0$ is a unit in $R[[X]]$ since $a_0 d_0 - b_0 c_0$ is a unit in R. Therefore, $R[[X]]$ is an Hermite ing.

Following Estes and Ohm [10], we say that a commutative ring R has 1 *in its stable range* if whenever $(a_1, a_2, \ldots, a_n) = R$ for $a_i \in R$, there exist $b_2, \ldots, b_n \in R$ such that $a_1 + b_2 a_2 + \cdots + b_n a_n$ is a unit. Note that R has 1 in its stable range if and only if for all $a_1, a_2 \in R$ with $(a_1, a_2) = R$ there exists $b_2 \in R$ such that $a_1 + b_2 a_2$ is a unit.

LEMMA 2. *Suppose R has 1 in its stable range. Then $R[[X]]$ has 1 in its stable range.*

Proof. Suppose $f, g \in R[[X]]$ and $(f, g) = R[[X]]$. Let f_0 (resp. g_0) be the constant term of f (resp. g). Then $(f_0, g_0) = R$ and so there exists $b \in R$ such that $f_0 + bg_0$ is a unit in R. It follows that $f + bg$ is a unit in $R[[X]]$.

LEMMA 3. *Let R be a von Neumann regular ring. Then R has 1 in its stable range. Consequently, $R[[X]]$ has 1 in its stable range.*

Proof. By Lemma 2, we have only to prove the first assertion. Suppose $a, b \in R$ and $(a, b) = R$. Then, there exist $s, t \in R$ such that $sa + tb = 1$. Since R is von Neumann regular, there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a = eu$. Thus, $1 - e = (1 - e)(sa + tb) = (1 - e)\, tb$ and $e + (1 - e)\, tb = 1$. Therefore, $a + u(1 - e)\, tb = u$, completing the proof.

We now come to the principal result of this subsection.

THEOREM 4. *For a commutative ring R, the following conditions are equivalent and each implies that R is a von Neumann regular ring.*

(a) *$R[[X]]$ is an elementary divisor ring.*

(b) *$R[[X]]$ is an Hermite ring.*

(c) *$R[[X]]$ is a Bézout ring.*

Proof. By virtue of Lemma 1 and Proposition 2, we have only to prove that (b) implies (a). In view of Lemmas 1 and 3, the proof will be finished once we prove

LEMMA 4. *Let S be an Hermite ring with 1 in its stable range. Then S is an elementary divisor ring.*

Proof. By the results in [16], it suffices to prove that $A = \left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right]$ can be diagonalized, where $(a, b, c) = S$. Now, 1 in the stable range of S implies that there exist elements $r, s \in S$ such that $b + ra + sc = u$, a unit. Set

$$P = \begin{bmatrix} 1 & s \\ -cu^{-1} & 1 - csu^{-1} \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} r & 1 - rau^{-1} \\ 1 & -au^{-1} \end{bmatrix}.$$

Note that P and Q are invertible and PAQ is diagonal.

Remark. In [5] examples are given of von Neumann regular rings R such that $R[[X]]$ is not a Bézout ring.

3. The FC-Property

As we observed earlier, the FC-property is a strong form of pole assignability. In the classical situation, when the entries of the system belong to a field, the FC-property holds. In fact any semi-quasi-local ring has the FC-property [1]. On the other hand, the ring of integers does not, nor does $\mathbb{R}[X]$, $\mathbb{R}$ the real field. If $\mathbb{C}$ denotes the complex field, it is an open problem to determine whether or not $\mathbb{C}[X]$ has the FC-property.

In this section we prove a theorem and give some examples that shed light on the problem of determining those rings that have the FC-property. We begin with the main theorem of this section.

Theorem 5. *Let R be a commutative ring with* 1 *in its stable range. Then R has the FC-property if and only if R has the GCU-property.*

The proof involves a succession of technical lemmas.

Lemma 5. *Suppose R has* 1 *in its stable range and G is an $n \times m$ matrix over R with a unimodular in its image. Then, there exist an $n \times n$ invertible matrix A and an $m \times m$ invertible matrix B such that*

$$AGB = \left[\begin{array}{ccc|c} & & & 0 \\ & & & \vdots \\ & * & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right].$$

Proof. By hypothesis, there exist $s_1, \ldots, s_m$, $u_1, \ldots, u_n \in R$ such that

$$G \begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \qquad \text{and} \qquad (u_1, \ldots, u_n) = R.$$

Since R has 1 in this stable range, there exist $r_1, \ldots, r_{n-1} \in R$ such that $r_1 u_1 + \cdots + r_{n-1} u_{n-1} + u_n = u$, a unit. Set

$$A_1 = \left[\begin{array}{ccc|c} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ \hline r_1 & \cdots & r_{n-1} & 1 \end{array}\right]$$

and note that

$$A_1 G \begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix} = A_1 \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_{n-1} \\ u \end{bmatrix}.$$

Thus, the last row of $A_1 G$ is unimodular. Say

$$A_1 G = \left[\begin{array}{ccc} & * & \\ \hline a_1 & \cdots & a_m \end{array}\right].$$

Since 1 is in the stable range of R, there exist $t_1, \ldots, t_{m-1}$ such that $t_1 a_1 + \cdots + t_{m-1} a_{m-1} + a_m = v$, a unit. Set

$$B_1 = \left[\begin{array}{ccc|c} 1 & & 0 & t_1 \\ & \ddots & & \vdots \\ 0 & & 1 & t_{m-1} \\ \hline 0 & \cdots & 0 & 1 \end{array}\right]$$

and note that

$$A_1 G B_1 = \left[\begin{array}{cccc} & & * & \\ a_1 & \cdots & a_{m-1} & v \end{array}\right].$$

Set

$$A_2 = \left[\begin{array}{ccc|c} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ \hline 0 & \cdots & 0 & v^{-1} \end{array}\right]$$

and note that $A_2 A_1 G B_1$ is of the form

$$\left[\begin{array}{ccc|c} & & & * \\ & * & & \vdots \\ & & & * \\ \hline * & \cdots & * & 1 \end{array}\right].$$

Finally, it is clear that there exist invertible A_3 and B_2 such that

$$A_3A_2A_1GB_1B_2=\left[\begin{array}{ccc|c} & * & & 0 \\ & & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right].$$

Set $A=A_3A_2A_1$ and $B=B_1B_2$.

Before continuing we need a definition. Let (F, G) be a system over R. Call a system $(\tilde{F}, \tilde{G})$ *systems equivalent* to (F, G), and write $(F, G)\sim(\tilde{F}, \tilde{G})$, if it is obtained from (F, G) by one of the following three transformations:

(i) $F\mapsto\tilde{F}=AFA^{-1}$, $G\mapsto\tilde{G}=AG$, for invertible A;

(ii) $F\mapsto\tilde{F}=F+GK$, $G\mapsto\tilde{G}=G$, for any K of suitable size; and

(iii) $F\mapsto\tilde{F}=F$, $G\mapsto\tilde{G}=GB$, for invertible B.

It is clear that systems equivalence is an equivalence relation. Moreover, if $(F, G)\sim(\tilde{F}, \tilde{G})$, then (F, G) is reachable (resp. feeds back to a cyclic vector) if and only if $(\tilde{F}, \tilde{G})$ is reachable (resp. feeds back to a cyclic vector).

COROLLARY 1. *Suppose R has 1 in its stable range and the GCU-property. Then any reachable system over R is systems equivalent to one of the form*

$$\left(\left[\begin{array}{cccc} & * & & \\ & & & \\ \hline 0 & 0 & \cdots & 0 \end{array}\right], \left[\begin{array}{ccc|c} & * & & 0 \\ & & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right]\right).$$

Proof. Given a reachable system (F, G) select A and B as in Lemma 5 so that

$$AGB=\left[\begin{array}{ccc|c} & * & & 0 \\ & & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right].$$

Then $(F, G)\sim(\tilde{F}, \tilde{G})=(AFA^{-1}, AGB)$, and $(\tilde{F}, \tilde{G})$ can be put in the desired form by a suitable feedback—just replace $(\tilde{F}, \tilde{G})$ by the system $(\tilde{F}+\tilde{G}K, \tilde{G})$, where

$$K=\left[\begin{array}{ccc} & 0 & \\ \hline -a_1 & \cdots & -a_n \end{array}\right] \quad \text{and} \quad [a_1 \cdots a_n] \quad \text{is the last row of } \tilde{F}.$$

LEMMA 6. *Suppose F_1, G_1, G_2, K_1, K_2 are matrices over R of sizes $(n-1)\times(n-1)$, $(n-1)\times 1$, $(n-1)\times(m-1)$, $1\times(n-1)$ and $(m-1)\times(n-1)$, respectively, $(n, m \geqslant 2)$. Set*

$$F=\left[\begin{array}{c|c} F_1 & G_1 \\ \hline 0 \cdots 0 & 0 \end{array}\right] \quad \text{and} \quad G=\left[\begin{array}{c|c} G_2 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 1 \end{array}\right].$$

Then,

$$(F, G) \sim \left(\left[\begin{array}{c|c} F_1+G_1K_1+G_2K_2 & G_1 \\ \hline 0 \cdots 0 & 0 \end{array}\right], G\right).$$

Proof.

$$(F, G) \sim \left(\left[\begin{array}{c|c} I & 0 \\ \hline -K_1 & 1 \end{array}\right] \left[\begin{array}{c|c} F_1 & G_1 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} I & 0 \\ \hline K_1 & 1 \end{array}\right],\right.$$

$$\left.\left[\begin{array}{c|c} I & 0 \\ \hline -K_1 & 1 \end{array}\right] \left[\begin{array}{c|c} G_2 & 0 \\ \hline 0 & 1 \end{array}\right]\right)$$

$$= \left(\left[\begin{array}{c|c} F_1+G_1K_1 & G_1 \\ \hline * & * \end{array}\right], \left[\begin{array}{c|c} G_2 & 0 \\ \hline -K_1G_2 & 1 \end{array}\right]\right)$$

$$\sim \left(\left[\begin{array}{c|c} F_1+G_1K_1 & G_1 \\ \hline * & * \end{array}\right] + \left[\begin{array}{c|c} G_2 & 0 \\ \hline -K_1G_2 & 1 \end{array}\right] \left[\begin{array}{c|c} K_2 & 0 \\ \hline 0 & 0 \end{array}\right], \left[\begin{array}{c|c} G_2 & 0 \\ \hline -K_1G_2 & 1 \end{array}\right]\right)$$

$$= \left(\left[\begin{array}{c|c} F_1+G_1K_1+G_2K_2 & G_1 \\ \hline * & * \end{array}\right], \left[\begin{array}{c|c} G_2 & 0 \\ \hline -K_1G_2 & 1 \end{array}\right]\right)$$

$$\sim \left(\left[\begin{array}{c|c} F_1 + G_1K_1 + G_2K_2 & G_1 \\ \hline * & * \end{array}\right], \left[\begin{array}{c|c} G_2 & 0 \\ \hline -K_1G_2 & 1 \end{array}\right] \left[\begin{array}{c|c} I & 0 \\ \hline K_1G_2 & 1 \end{array}\right] \right)$$

$$= \left(\left[\begin{array}{c|c} F_1 + G_1K_1 + G_2K_2 & G_1 \\ \hline * & * \end{array}\right], G \right).$$

A suitable feedback completes the proof.

LEMMA 7. *Suppose the $n \times n$ matrix F has a cyclic vector $u \in R^n$. Then, there exist an $n \times n$ invertible matrix A and a $1 \times n$ matrix K such that*

$$A^{-1}FA + A^{-1}uK = \left[\begin{array}{ccccc} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & & \ddots & \\ & 0 & & 0 & 1 \\ \hline 0 & 0 & \cdots & & 0 \end{array}\right].$$

Proof. By hypothesis, the matrix $S = [u, Fu, \ldots, F^{n-1}u]$ is invertible. Note that

$$S^{-1}FS = \left[\begin{array}{cccc|c} 0 & & & 0 & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & & & \vdots \\ & & \ddots & 0 & \\ & 0 & & 1 & -a_{n-1} \end{array}\right],$$

where $X^n + a_{n-1}X^{n-1} + \cdots + a_0$ is the characteristic polynomial of F. Note further that

$$S^{-1}u = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Set

$$T = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & & 1 & \\ \vdots & \vdots & & & \\ & a_{n-1} & & & 0 \\ a_{n-1} & 1 & & & \\ 1 & & & & \end{bmatrix}$$

and note that T^{-1} exists. Moreover,

$$T^{-1}S^{-1}FST=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \hline & & \ddots & \ddots & \\ & 0 & & 1 & 0 \\ & & & & 1 \\ \hline -a_0 & -a_1 & & \cdots & -a_{n-1} \end{bmatrix} \quad \text{and} \quad T^{-1}S^{-1}u=\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Set $A=ST$ and $K=[a_0\, a_1 \cdots a_{n-1}]$. Then

$$A^{-1}FA+A^{-1}uK=\begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & 0 & \ddots & \ddots & \\ & & & 0 & 1 \\ \hline 0 & 0 & & \cdots & 0 \end{bmatrix}, \quad \text{as required.}$$

We are now in a position to prove Theorem 5. If R has the FC-property, it is clear from the definitions that R has the GCU-property (regardless of R having 1 in its stable range).

Conversely, suppose R has 1 in its stable range and has the GCU-property. Let (F, G) be a reachable system over R with F $n\times n$ and G $n\times m$. We proceed by induction on n. Since the case $n=1$ is trivial, we may assume that $n>1$ and that the Theorem holds for all positive integers $k<n$. Moreover, we may assume $m>1$ (otherwise there is nothing to prove).

By Corollary 1, we may assume that

$$(F, G)=\left(\left[\begin{array}{ccc|c} & F_1 & & G_1 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} & & & 0 \\ & & & \vdots \\ & G_2 & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right]\right),$$

where F_1 and G_2 are of sizes $(n-1)\times(n-1)$ and $(n-1)\times(m-1)$, respectively, and $G_1\in R^{n-1}$. By Lemma 1 in Eising [8], the system $(F_1, [G_1, G_2])$ is reachable. Therefore, by induction, there is vector $v\in R^m$ and an $m\times(n-1)$ matrix K such that $[G_1, G_2]\,v$ is cyclic vector for $F_1+[G_1, G_2]\,K$. Write $K=[K_1 \mid K_2]$ where K_1 is $1\times(n-1)$ and K_2 is $(m-1)\times(n-1)$. Then $F_1+[G_1, G_2]\,K=F_1+G_1K_1+G_2K_2$. By Lemma 6, (F, G) is systems equivalent to

$$\left(\left[\begin{array}{ccc|c} & \tilde{F} & & G_1 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right], G\right),$$

where $\tilde{F} = F_1 + G_1 K_1 + G_2 K_2$ has a cyclic vector $u = [G_1, G_2]\, v \in R^{n-1}$. By Lemma 7, there exist an $(n-1)\times(n-1)$ invertible matrix A, and a $1\times(n-1)$ matrix $\tilde{K}$ such that

$$A^{-1}\tilde{F}A + A^{-1}u\tilde{K} = \left[\begin{array}{ccccc} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & 0 & & 0 & 1 \\ \hline 0 & & \cdots & & 0 \end{array}\right].$$

Note that

$$\left[\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} \tilde{F} & G_1 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} A^{-1}\tilde{F}A & A^{-1}G_1 \\ \hline 0 & 0 \end{array}\right]$$

and

$$\left[\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} G_2 & 0 \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} A^{-1}G_2 & 0 \\ \hline 0 & 1 \end{array}\right].$$

So,

$$(F, G) \sim \left(\left[\begin{array}{c|c} A^{-1}\tilde{F}A & A^{-1}G_1 \\ \hline 0 & 0 \end{array}\right], \left[\begin{array}{c|c} A^{-1}G_2 & 0 \\ \hline 0 & 1 \end{array}\right]\right).$$

Write $v\tilde{K} = [\tilde{K}_1 \,|\, \tilde{K}_2]$, where $\tilde{K}_1$ is $1\times(n-1)$ and $\tilde{K}_2$ is $(m-1)\times(n-1)$. Then, $A^{-1}u\tilde{K} = A^{-1}[G_1, G_2]\, v\tilde{K} = A^{-1}G_1\tilde{K}_1 + A^{-1}G_2\tilde{K}_2$. By Lemma 6,

$$(F, G)$$

$$\sim \left(\left[\begin{array}{c|c} A^{-1}\tilde{F}A + A^{-1}G_1\tilde{K}_1 + A^{-1}G_2\tilde{K}_2 & A^{-1}G_1 \\ \hline 0 & 0 \end{array}\right], \left[\begin{array}{c|c} A^{-1}G_2 & 0 \\ \hline 0 & 1 \end{array}\right]\right)$$

$$= \left(\left[\begin{array}{c|c} A^{-1}\tilde{F}A + A^{-1}u\tilde{K} & A^{-1}G_1 \\ \hline 0 & 0 \end{array}\right], \left[\begin{array}{c|c} A^{-1}G_2 & 0 \\ \hline 0 & 1 \end{array}\right]\right)$$

Write

$$A^{-1}G_1 = \begin{bmatrix} r_1 \\ \vdots \\ r_{n-1} \end{bmatrix}, \qquad A^{-1}G_2 = \begin{bmatrix} * \\ \hline s_1 \cdots s_{m-1} \end{bmatrix}.$$

By Lemma 1 in [8],

$$(A^{-1}\tilde{F}A + A^{-1}u\tilde{K}, [A^{-1}G_1, A^{-1}G_2])$$
$$= \left(\begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & & \ddots & \\ & 0 & & 0 & 1 \\ \hline 0 & & \cdots & & 0 \end{bmatrix}, \begin{bmatrix} r_1 & & & \\ \vdots & & * & \\ r_{n-1} & s_1 & \cdots & s_{m-1} \end{bmatrix} \right)$$

is reachable. This implies that $(r_{n-1}, s_1, \ldots, s_{m-1}) = R$. Since R has 1 in its stable range, there exist $k_1, \ldots, k_{m-1} \in R$ such that $r_{n-1} + k_1 s_1 + \cdots + k_{m-1} s_{m-1} = r$, a unit. Set

$$K' = \left[\begin{array}{c|c} & k_1 \\ & \vdots \\ 0 & k_{m-1} \\ \hline 0 \quad \cdots & 0 \end{array}\right], \qquad \text{an } m \times n \text{ matrix.}$$

Then,

$$\left[\begin{array}{c|c} A^{-1}\tilde{F}A + A^{-1}u\tilde{K} & A^{-1}G_1 \\ \hline 0 & 0 \end{array}\right] + \left[\begin{array}{c|c} A^{-1}G_2 & 0 \\ \hline 0 & 1 \end{array}\right] K'$$
$$= \left[\begin{array}{ccccc|c} 0 & 1 & & & 0 & * \\ & 0 & 1 & & & * \\ & & & \ddots & & \\ & 0 & & 0 & 1 & * \\ \hline 0 & 0 & \cdots & & 0 & r \\ 0 & 0 & \cdots & & 0 & 0 \end{array}\right], \qquad \text{which has the cyclic vector } \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

in the image of

$$\left[\begin{array}{c|c} A^{-1}G_2 & 0 \\ \hline 0 & 1 \end{array}\right].$$

This completes the proof of Theorem 5.

COROLLARY 2. *Let R be an Hermite ring with* 1 *in its stable range. Then R has the FC-property. In particular, a Bézout domain with* 1 *in its stable range has the FC-property.*

Proof. By Lemma 4, R is an elementary divisor ring. As noted earlier on, an elementary divisor ring has the *UCU*-property and, a fortiori, the *GCU*-property. The result follows from Theorem 5 and the fact that any Bézout domain is an Hermite ring [16].

Remark. Even though the ring of integers $\mathbb{Z}$ is a Bézout domain and has the *GCU*-property, it is known that $\mathbb{Z}$ does not have the *FC*-property [6]. This shows that having 1 in the stable range cannot be deleted from the statements of Theorem 5 and Corollary 2.

To apply Theorem 5 to construct examples of rings having the *FC*-property, we recall the definition of a class of rings which may not be familiar to all our readers.

A commutative ring R is said to be a *local-global* ring if each polynomial over R (in several variables) admitting unit values locally, admits unit values. (See [9] and [18] for more detail.)

PROPOSITION 3. *A local-global ring has* 1 *in its stable range as well as the GCU-property. In particular, local-global rings have the FC-property.*

Proof. Local-global rings clearly have 1 in their stable range since quasi-local rings do. Next, let G be an $n \times m$ matrix, let $X_1,\dots, X_n$, $Y_1,\dots, Y_m$ be indeterminates over R, and set

$$f(X_1,\dots, X_n, Y_1,\dots, Y_m) = [X_1 \cdots X_n]\; G \begin{bmatrix} Y_1 \\ \vdots \\ Y_m \end{bmatrix}.$$

Then G has a unimodular vector in its image if and only if f has a unit value. If G is good, then f admits unit values locally since quasi-local rings clearly have the *GCU*-property. Thus, by the local-global property, f has a unit value. The last assertion follows immediately from Theorem 5.

So, one way to give examples of rings having the *FC*-property is to give examples of local-global rings. Some well-known local global rings are: semi-quasi-local rings, direct products of local-global rings, zero-dimensional rings, and the ring of all algebraic integers. Another known example which will be of special interest to us is the following.

EXAMPLE. Let R be a commutative ring with X an indeterminate. Let $S = \{f \in R[X] \mid \text{the content of } f \text{ is } R\}$. Then $R(X) = (R[X])_S$ is a local-global ring. The proof is easy, but we include it anyway.

Proof. Let $T = R(X)$. To show that T is a local-global ring, it suffices to show that each polynomial in several variables over T with unit content has a unit value (since any polynomial with unit values locally has unit content). By the lemmas in [18], it suffices to consider polynomials in only one variable over T. Thus let $f(Y) = \sum_{i=0}^{n} (a_i/b_i) Y^i$ where $a_i \in R[X]$, $b_i \in S$, and Y is an indeterminate over T. If $f(Y)$ has unit content over T, there exists $b \in S$ such that $(a_0, \ldots, a_n) R[X] = bR[X]$. Thus, the coefficients of the a_i together generate R. If we let $d = \max\{\deg a_i\} + 1$, then $a_0 + a_1 X^d + \cdots + a_n X^{nd} \in S$. Therefore, $f(X^d)$ is a unit in T.

Now, $R(X)$ has been a much studied ring in other contexts and a great deal is known about the relationship between the ideal structures of R and $R(X)$; see [11, p. 410], for example. For many clases of rings, R belongs to the class if and only if $R(X)$ does; and moreover, the map $M \mapsto M \cdot R(X)$ is a homeomorphism from the maximal spectrum of R to that of $R(X)$.

In summary, these remarks show that just about any type of ring (e.g., *UFD*, integrally closed, regular, non-semi-quasi-local, etc.) can have the *FC*-property. Hence, the *FC*-property appears to be neither ideal-theoretic nor topological.

4. Prüfer Domains with the *UCS*-Property

The motivation for this section comes from two sources. As we saw in the previous section, a Bézout domain with 1 in its stable range has the *FC*-property while the ring of integers does not have the *FC*-property. On the other hand, being an elementary divisor domain, $\mathbb{Z}$ does have the *PA*-property. It is an old open question whether or not each Bézout domain is an elementary divisor domain, but as noted in [4], a Bézout domain is an elementary divisor domain if and only if it has the *UCU*-property. (Over a Bézout domain, rank one projective modules are free and so the *UCU* and *UCS* properties are equivalent.) But a Bézout domain is an elementary divisor domain if and only if it has the "Simultaneous Basis Property" and so for Bézout domains the *UCU*-property is equivalent to the Simultaneous Basis Property. In this section we will formulate a notion of Simultaneous Basis Property for Prüfer domains and prove that a Prüfer domain D has it if and only if D has the UCS-property. This will give a Prüferized version of elementary divisor domains and lead to a variation of the old problem:

QUESTION. Does every Prüfer domain have the UCS-property?

A second motivation for this question comes from a beautiful theorem of Hautus and Sontag. In [15], they prove that a Dedekind domain has the PA-property by showing that every Dedekind domain has the UCS-property. Thus, from the pole assignability standpoint, it is interesting to try to show that Prüfer domains have the UCS-property.

We have no counterexample to the question mentioned above, but believe that there must be one. A better problem is to determine which Prüfer domains have the UCS-property.

The first of our results deals with the UCU-property.

LEMMA 8, *Let R be a ring having the UCU-property. Then every homomorphic image S of R has the property that projective S-modules of constant finite rank are free.*

Proof. By Theorem 1 it suffices to show that projective R-modules of constant finite rank are free. By the UCU-property and induction on the rank, it suffices to show that for all $n \geqslant 1$, any locally nonzero summand of R^n is the submodule generated by the columns of some unit content matrix. Let $n \geqslant 1$ and suppose $R^n = A \oplus K$ where A is locally nonzero. Select column vectors $v_1, \dots, v_m \in R^n$ generating A, and let G be the matrix $[v_1 \cdots v_m]$. If G does not have unit content, there exists a maximal ideal M of R containing the entries of G. Hence $A \subseteq MR^n$ and $R^n = K + MR^n$. Since K is finitely generated, Nakayama's Lemma implies that $K_M = R^n_M$. Thus $A_M = 0$, a contradiction.

PROPOSITION 4. *Let D be a Prüfer domain. Then D has the UCU-property if and only if D is an elementary divisor domain. In particular, if D is a Dedekind domain, then D has the UCU-property if and only if D is a* PID.

Proof. If D has the UCU-property, then, by Lemma 8, each finitely generated ideal of D is free; that is, a principal ideal. Hence, D is a Bézout domain. But a Bézout domain having the UCU-property is known to be an elementary divisor domain [4]. That elementary divisor domains have the UCU-property is also well known [6].

Remark. It is clear from Proposition 4 that a Dedekind domain D with nontrivial 2-torsion in its class group does not have the UCU-property. We show, in fact, that it does not even have the weaker GCU-property: Suppose I is a nonprincipal ideal of D with principal square. It is known that $I \oplus I \cong D^2$ as D-modules. Thus I can be identified with the submodule of D^2 generated by the columns of a 2×2 matrix G, and the proof of Theorem 2 in [3] shows that G is good. If D had the GCU-property, the image of G would contain a unimodular vector and be free. Hence, I would be free and therefore principal, a contradiction.

In [20] it is shown that if R is a ring with stably free modules free and rank 1 projectives free, then the GCU and PA-properties are equivalent. For D as above, D has stably frees free and the PA-property, but not the GCU-property. Hence, the assumption on rank 1 projectives cannot be deleted.

We return to the main focus of this section. Let D be a Bézout domain with M a finitely generated submodule of D^n. Since D is a Bézout domain, M is a free D-module, say of rank k. By a *simultaneous basis* for M and D^n we mean a basis $\{x_1,..., x_n\}$ for D^n and ring elements $d_1,..., d_k$ such that $\{d_1 x_1,..., d_k x_k\}$ is a basis for M. We say that D has the *Simultaneous Basis Property* if and only if for each positive integer n, D^n and each finitely generated submodule M of D^n have a simultaneous basis.

As I. Kaplansky pointed out to us, a Bézout domain has the Simultaneous Basis Property if and only if it is an elementary divisor domain. (The verification of this is straightforward but nonilluminating; so, we omit it.) Thus, for a Bézout domain D the following are equivalent: D is an elementary divisor domain; D has the Simultaneous Basis Property; D has the UCU-property. We proceed to define an analogous "Simultaneous Basis Property" for Prüfer domains and show that it is equivalent to the UCS-property.

Let D be a Prüfer domain with M a finitely generated submodule of D^n. Since D is a Prüfer domain, M is a projective D-module, say of constant rank k. By a *simultaneous basis* for M and D^n we mean a collection of rank 1 projective submodules $P_1,..., P_n \subseteq D^n$ and ideals $I_1 \supseteq \cdots \supseteq I_k$ of D such that $D^n = P_1 \oplus \cdots \oplus P_n$ and $M = I_1 P_1 \oplus \cdots \oplus I_k P_k$. We say that D has the *Simultaneous Basis Property* if and only if for each positive integer n, D^n and each finitely generated submodule M of D^n have a simultaneous basis.

Remark. It is known that Dedekind domains have the Simultaneous Basis Property (see [7, Theorem 22.12]).

Remark. A finitely generated submodule M of D^n is said to have unit content if and only if any matrix whose columns generate M has unit content. Using the characterization of rank 1 projective modules given in [12], it is easily checked that D has the UCS-property if and only if each finitely generated submodule of D^n with unit content contains a rank 1 projective summand of D^n.

THEOREM 6. *Let D be a Prüfer domain. Then, D has the UCS-property if and only if D has the Simultaneous Basis Property.*

Proof. First suppose D has the UCS-property and let M be a nonzero finitely generated submodule of D^n $(n \geqslant 1)$. If K is any finitely generated submodule of D^n, let $c(K)$ denote the content of K; that is the content of any matrix whose columns generate K. Set $J_1 = c(M)$. Since D is a Prüfer domain, J_1 is invertible and $M = J_1(J_1^{-1}M)$. Since $J_1^{-1}M$ is a finitely generated submodule of D^n with unit content, the UCS-property yields a rank 1 projective summand P_1 of D^n contained in $J_1^{-1}M$. Write $D^n = P_1 \oplus N_1$. Thus $J_1^{-1}M = P_1 \oplus M_1$ (where $M_1 = N_1 \cap J_1^{-1}M$ and $M = J_1P_1 \oplus J_1M_1$. If $J_1M_1 \neq 0$, set $J_2 = c(M_1)$. As above, J_2 is invertible and $J_2^{-1}M_1$ contains a rank 1 projective summand P_2 of D^n. Therefore, $D^n = P_1 \oplus P_2 \oplus N_2$ and $M = J_1P_1 \oplus J_1J_2P_2 \oplus J_1J_2M_2$ where $M_2 = N_2 \cap J_2^{-1}M_1$. Continuing in this way, we eventually reach $k \leqslant n$ with $J_1J_2 \cdots J_kM_k = 0$, $D^n = P_1 \oplus \cdots \oplus P_k \oplus N_k$, and $M = J_1P_1 \oplus \cdots \oplus J_1J_2 \cdots J_kP_k$. Setting $I_r = J_1J_2 \cdots J_r$ $(1 \leqslant r \leqslant k)$, and observing that (since D is a Prüfer domain) N_k is zero or a direct sum of rank 1 projectives, we conclude that D has the Simultaneous Basis Property.

Conversely, suppose D has the Simultaneous Basis Property and let G be an $n \times m$ matrix over D with unit content. Let $M \subseteq D^n$ be the submodule generated by the columns of G. Thus, there exist rank 1 projective summands $P_1, \ldots, P_k$ of D^n and ideals $I_1 \supseteq \cdots \supseteq I_k$ of D such that $M = I_1P_1 \oplus \cdots \oplus I_kP_k$. Since M has unit content, $I_1 = R$ and so $I_1P_1 = P_1$. Therefore M contains a rank 1 projective summand of D^n and we conclude that D has the UCS-property.

We close by indicating how to obtain non-Noetherian, non-semi-quasi-local Prüfer domains with the UCS-property. One way to do this is to use a nice theorem of Levy [17] which we now present. Let D be a one-dimensional Prüfer domain with $\{M_i\}$ the family of all maximal ideals of D. We say that D has *finite character* if each nonzero element of D belongs to only finitely many M_i's. The following theorem extends Corollary 22.14 of [7].

THEOREM. *Let D be a one-dimensional Prüfer domain of finite character. Then D has the UCS-property.*

We will not give a proof of the theorem since it is to appear elsewhere. We should add that in [14, Proposition 1.2], Heinzer shows how to actually construct (non-Noetherian) domains of the type referred to in Levy's theorem. Moreover, the domains constructed in [14] are not Bézout domains.

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