May 6: Mixed Multiplicities and Teissier's Theorem, part 2

We now show the existence of the Hilbert-Samuel polynomial associated to two $\mathfrak{m}$-primary ideals.

Theorem D6. Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d, and $I, J \subseteq R$ $\mathfrak{m}$-primary ideals. Then there exists a numerical polynomial $P_{I,J}(n, m)$ of degree d such that $\lambda(R/I^nJ^m) = P_{I,J}(n, m)$, for $n, m >> 0$. Moreover, if we write the terms of total degree d in $P_{I,J}(n, m)$ as

$$\frac{1}{d!}\{e_0(I|J)n^d + \binom{d}{1} e_1(I|J)n^{d-1}m + \cdots + \binom{d}{d-1} e_{d-1}(I|J)nm^{d-1} + e_d(I|J)m^d\},$$

then each $e_i(I|J) > 0$.

Proof. We use $H_{I,J}(n, m)$ to denote $\lambda(R/I^nJ^m)$ for all n, m. Without loss of generality, we may assume the residue field of R is infinite.We now induct on d. If $\dim(R) = 0$, then the conclusion of the theorem is clear.

Assume $d > 0$. By (x) in the General Discussion above, and its predecessor in the previous section, if we write $S := L$, for $L := (0 : \mathfrak{m}^t)$, for $t >> 0$, the lengths $H_{I,J}(n, m)$ and the lengths $\lambda(S/I^nJ^mS)$ differ by a constant for large n, m. It follows that we may assume that R has positive depth, and hence I and J have positive grade.

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Thus, there is $a \in I$, a non-zero divisor, such that $(I^n J^m : a) = I^{n-1}J^m$, for all $n > c$ and $m \geq 1$. Set $R^* := R/aR$. For $n > c$, the exact sequence

$$0 \to R/I^{n-1}J^m \xrightarrow{\cdot a} R/I^n J^m \to R^*/I^n J^m R^* \to 0,$$

gives

$$H_{I,J}(n, m) - H_{I,J}(n-1, m) = \lambda(R^*/I^n J^m R^*).$$

Since $\dim(R^*) = d - 1$, by induction on d, the lengths of $R^*/I^n J^m R^*$ agree with a polynomial of degree $d - 1$, all of whose top coefficients are positive.

Thus, $H_{I,J}(n, m)$ agrees with a polynomial numerical polynomial $P_{I,J}(n, m)$ of degree d, for $n, m >> 0$.

If we write the terms of degree d in $P_{I,J}(n, m)$ as in the statement of the theorem, item (viii) in the discussion above shows $e_i(I|J) = e_i(IR^*|JR^*) > 0$, for $0 \leq i \leq d - 1$.

Since $e_d(I|J) = e(J) > 0$ (by part (xiii)), the proof is complete. □

A special case of Lech's Lemma

Proposition D6. Let $(R, \mathfrak{m}, k)$ be a two-dimensional local ring with infinite residue and $I = (a, b)R$ an ideal generated by a system of parameters. Assume that $a \in I$ is a non-zerodivisor and a superficial element for I. Then,

$$e(I) = \lim_{n\to\infty} \frac{1}{n^2} \cdot \lambda(R/(a^n, b^n)R).$$

Proof. Let's first note that $e(I^n) = e((a^n, b^n)R)$, for all n. To see this, let $a^i b^j \in I^n$ be a monomial generator of degree n. Then

$$(a^i b^j)^n = (a^n)^i (b^n)^j \in (a^n, b^n)^n R.$$

This shows $a^i b^j$ is integral over (a^n, b^n), and thus I^n and $(a^n, b^n)R$ have the same integral closure, and thus, the same multiplicity. Therefore,

$$n^2 e(I) = e(I^n) = e((a^n, b^n)R) \leq \lambda(R/(a^n, b^n)R) \leq n \cdot \lambda(R/(a, b^n)R).$$

Here we are using the fact that if $I \subseteq R$ is an $\mathfrak{m}$-primary ideal in a local ring of dimension d, then for any $r \geq 1$, $e(I^r) = r^d e(I)$.

A special case of Lech's Lemma

Dividing the displayed equation by n^2 and taking the limit as $n \to 0$ we have,

$$\begin{aligned} e(I) &\leq \lim_{n\to\infty} \frac{1}{n^2} \cdot \lambda(R/(a^n, b^n)R) \\ &\leq \lim_{n\to\infty} \frac{1}{n} \cdot \lambda(R/(a, b^n)R) \\ &= e(b, R/aR) \\ &= e(I/aR) \\ &= e(I), \end{aligned}$$

since $a \in I$ is a superficial, non-zerodivisor.

The key step

Here is the main result in the Rees-Sharp proof.

Theorem E6. Let $(R, \mathfrak{m}, k)$ be a two-dimensional local ring and $I, J \subseteq R$ $\mathfrak{m}$-primary ideals. Then,

$$e(IJ) \leq 2e(I) + 2e(J).$$

Proof. We may assume that the residue field of R is infinite. By modding out the stable value of $(0 : \mathfrak{m}^t)$, we may assume I, J have positive grade.

By Theorem F5, there exists an ideal $K \subseteq I$ generated by a system of parameters with $\overline{K} = \overline{I}$. By the comments following Theorem O5, $e(K) = e(I)$.

On the other hand, one also has $\overline{KJ} = \overline{IJ}$, and thus, $e(KJ) = e(IJ)$.

Therefore, we may replace I by K, then change notation to assume that $I = (a, b)R$ is generated by a system of parameters.

From our work in the previous section, we may further assume that $I = (a, b)$ with a a non-zerodivisor and a superficial element for I.

The key step

Let $F = R^2$, and observe that for all $n \geq 1$, we have a surjective R-module map $F/J^nF \to (a^n, b^n)R/(a^n, b^n)J^n$. Thus,

$$2 \cdot \lambda(R/J^n) \geq \lambda\{(a^n, b^n)R/(a^n, b^n)J^n\}.$$

Therefore,

$$\begin{aligned}\lambda(R/(a^n, b^n)) + 2\lambda(R/J^n) &\geq \lambda(R/(a^n, b^n)) + \lambda\{(a^n, b^n)R/(a^n, b^n)J^n\} \\ &= \lambda(R/(a^n, b^n)J^n) \\ &\geq \lambda R/(I^nJ^n).\end{aligned}$$

If we multiply the left hand side of this inequality by $\frac{2}{n^2}$ and take the limit as $n \to \infty$, we get $2e(I) + 2e(J)$ (using Lech's lemma on the first term).

Multiplying the far right side of the inequality by $\frac{2}{n^2}$ and taking the limit as $n \to \infty$ gives, $e(IJ)$, which completes the proof. $\square$

Teissier's Theorem

We now have all of the pieces required to prove the main result of this section.

Proof of Theorem A6. We may assume the residue field of k is infinite, and proceed by induction on $d := \dim(R)$. We let $e_i(I|J)$ denote the mixed multiplicities of I and J, and set $e_i := e_i(I|J)$. We need to prove that $e_i^2 \leq e_{i-1}e_{i+1}$, for all $1 \leq i \leq d-1$.

Suppose $d = 2$. We must prove $e_1^2 \leq e_0e_2$. By item (iv) in the general discussion, for all $r, s \geq 1$, we have

$$e(I^rJ^s) = e_0r^2 + 2e_1rs + e_2s^2.$$

On the other hand, by Theorem E6,

$$e(I^rJ^s) \leq 2e(I^r) + 2e(J^s) = 2r^2e(I) + 2s^2e(J).$$

Thus,

$$e_0r^2 + 2e_1rs + e_2s^2 \leq 2e_0r^2 + 2e_2s^2,$$

for all $r, s \geq 1$. Therefore,

$$0 \leq e_0r^2 - 2e_1rs + e_2s^2,$$

for all r, s. Since $e_0 > 0$, if we substitute $r = e_1$ and $s = e_0$ into this last expression, we can divide by e_0 and conclude $e_1^2 \leq e_0e_2$.

Teissier's Theorem

Now suppose $d \geq 3$. By item (x) in the General Discussion, we may assume that I, J have positive grade. By Proposition C6, there exists $a \in I$, a non-zerodivisor that is superficial for the pair I, J.

Set $R^* := R/aR$, so $\dim(R^*) = d - 1$. Then for $n >> 0$, we have an exact sequence

$$R/I^{n-1}J^m \xrightarrow{\cdot a} R/I^nJ^m \to R^*/I^nJ^mR^* \to 0.$$

It follows that for $n, m >> 0$,

$$P_{I,J}(n, m) - P_{I,J}(n-1, m) = P_{IR^*,JR^*}(n, m).$$

By item (viii) in the General Discussion,

$$e_i(I|J) = e_i(IR^*, JR^*),$$

for $0 \leq i \leq d-1$. Therefore, by induction, we have $e_i^2 \leq e_{i-1}e_{i+1}$, for all $1 \leq i \leq d-2$.

Since the argument is symmetric in I and J, we may take $b \in J$ superficial for I, J and repeat what we have just done with the roles of I and J reversed to pick up the last relation $e_{d-1}^2 \leq e_{d-2}e_d$. □

Final Remarks. What about equality in the Minkowski inequality for multiplicities? It turns out that this is closely related to the integral closure of powers of ideals.

Two ideals $I, J \subseteq R$ are said to be *projectively equivalent* if there exist positive integers $a, b \geq 1$ such that $\overline{I^a} = \overline{J^b}$.

We first note that if this condition holds, then equality in the Minkowski inequality for multiplicities is more or less a formal consequence of the rules for manipulating the mixed multiplicities. To see this, we need to observe that if $L, K \subseteq R$ have the same integral closure, then $e_i(L|K) = e(L) = e(K)$, for all i.

To see this, for one, we know from the previous section that $e(L) = e(K)$. We may also assume $L = K$, by item (xii) in the General Discussion. So suppose $L = K$. Then when we calculate $P_{L,K}(n, m) = P_{L,L}(n, m)$ we are calculating the lengths of R/L^nK^m, with n, m independent. But this is the same as considering the lengths of R/L^{n+m} as a function of two variables. Thus, if expand $P_L(n + m)$ out as function of n, m and compare the leading coefficients with those of $P_{L,L}(n, m)$, we see that $e_i(L, L) = e(L)$, for all i.

Teissier's Theorem

Now, if we trace through the sequence of steps that led from the Minkowski inequality to the set of inequalities $e_i^2 \leq e_{i-1}e_{i+1}$, we see two things: (i) The Minkowski inequality holds if and only if the set of inequalities $e_i^2 \leq e_{i-1}e_{i+1}$ hold and (ii) Equality in the Minkowski inequality holds if and only if $e_i^2 = e_{i-1}e_{i+1}$, for all i.

Now, suppose $I, J \subseteq R$ are projectively equivalent, i.e., there exist $a, b \geq 1$ such that $\overline{I^a} = \overline{J^b}$. Then, all of the mixed multiplicities $e_i(\overline{I^a}|\overline{J^b})$ are equal, and consequently, all of the mixed multiplicities $e(I^a|J^b)$ are equal.

Thus $e_i(I^a|J^b)^2 = e_{i-1}(I^a|J^b)e_{i+1}(I^a|J^b)$, for all i. However, from item (iv) in the General Discussion we have,

$$e_i(I^a|J^b)^2 = (a^{d-i}b^i)^2 e_i(I|J)^2$$

and

$$e_{i-1}(I^a|J^b)e_{i+1}(I^a|J^b) = (d^{d-i+1}b^{i-1}e_{i-1}(I|J)) \cdot (a^{d-(i+1)}b^{i+1}e_{i+1}(I|J)),$$

from which it follows that $e_i(I|J)^2 = e_{i-1}(I|J)e_{i+1}(I|J)$, for all i, and thus, equality holds in the Minkowski inequality.

Teissier's Theorem

In the geometric setting, Teissier proved the converse, which turns out to be a generalization of the Rees multiplicity theorem.

In other works, the converse states that equality in the Minkowski inequality implies that the ideals are projectively equivalent.

Using geometric techniques, Teissier reduced the question to surfaces, and used resolutions of singularities to finish off the proof.

Rees and Sharp proved an algebraic version of this for two-dimensional quasi-unmixed local rings and DK showed how to reduced the general algebraic case to the two-dimensional case.

A consequence of this theorem is that one gets a version of the Rees multiplicity theorem, without assuming a containment relation between I and J.

Teissier's Theorem

The statement in this case would be:

Rees Multiplicity Theorem Generalized. Let $(R, \mathfrak{m}, k)$ be a quasi-unmixed local ring of dimension d and $I, J \subseteq R$, $\mathfrak{m}$-primary ideals. If

$$e_0(I|J) = e_1(I|J) = \cdots = e_d(I|J),$$

then $\overline{I} = \overline{J}$.

The point is that if all of the mixed multiplicities are equal, it is not hard to see that equality must hold in the Minkowski inequality. Thus, $\overline{I^a} = \overline{J^b}$, for some a, b.

But then $a^d e(I) = b^d e(J)$, and since $e(I) = e(J)$, $a = b$, so $\overline{I} = \overline{J}$.

The Rees Multiplicity Theorem has subsequently been generalized in a number of ways.