May 1: Mixed Multiplicities and Teissier's Theorem

Teissier's Theorem

The purpose of this section is to consider the following theorem of Teissier, which was given in a geometric setting.

Theorem A6. Let $(R, \mathfrak{m}, k)$ be a reduced local ring of dimension $d \geq 2$ such k has characteristic zero, R is Cohen-Macaulay and also the localization of a finitely generated k-algebra at a maximal ideal. Let $I, J \subseteq R$ be two $\mathfrak{m}$-primary ideals. Then:

$$e(IJ)^{\frac{1}{d}} \leq e(I)^{\frac{1}{d}} + e(J)^{\frac{1}{d}}.$$

This is similar in spirit to the Minkowski inequality from analysis which states that for $f, g \in L^p(\mathbb{R})$ (say),

$$(\int |f+g|^p \ dx)^{\frac{1}{p}} \leq (\int |f|^p \ dx)^{\frac{1}{p}} + (\int |g|^p \ dx)^{\frac{1}{p}}.$$

A few years after Teissier's result was published, Rees and Sharp wrote a paper extending the result to arbitrary local rings. We will present this result below.

Mixed Multiplicities

A key feature of Teissier's proof was the use of the so-called *mixed multiplicities* of I and J. These multiplicities are the normalized leading coefficients of the terms of total degree d in the Hilbert-Samuel polynomial that tracks the lengths of R/I^nJ^m, for $n, m >> 0$.

General Discussion. We begin with a general discussion of some of the things we will need for the Teissier-Rees-Sharp theorem.

(i) Just as in the one variable case, a numerical polynomial in two variables is a polynomial $P(x, y) \in \mathbb{Q}[x, y]$ such that $P(n, m) \in \mathbb{Z}$, for all $n, m \in \mathbb{Z}$ (or $\mathbb{N}$).

Given $\mathfrak{m}$-primary ideals $I, J \subseteq R$, there exists a numerical polynomial of degree d, $P_{I,J}(x, y) \in \mathbb{Q}[x, y]$, such that $P_{I,J}(n, m) = \lambda(R/I^nJ^m)$, for $n, m >> 0$.

We will prove this below, but the proof of this fact is not much different from the proof of the one variable case, though in our discussion we will not consider the general bi-graded case.

Henceforth, we will think of this polynomial as a polynomial in n and m.

(ii) Because $P_{I,J}(n,m)$ is a numerical polynomial, there exist integer coefficients, e_{ij} such that

$$P_{I,J}(n,m) = \sum_{i+j\leq d} e_{i,j} \binom{n+i}{i}\binom{m+j}{j}.$$

The proof of this is almost exactly the same as in the one variable case because one can induct on the degree of the second variable, and mimic the previous proof.

Moreover, using the identities $\binom{n+i}{i} = \binom{n+i+1}{i+1} - \binom{n+i-1}{i}$, the same proof from the previous section shows that if $H(n,m)$ is a numerical function, and $H(n,m) - H(n-1,m)$ agrees with a numerical polynomial of degree $d-1$ for $n, m >> 0$, then $H(n,m)$ agrees with a numerical polynomial of degree d, for $n, m >> 0$.

The integers e_{ij} displayed above such that $i+j=d$ are called the mixed multiplicities of I and J.

We will see below that the mixed multiplicities are positive integers.

Mixed Multiplicities

(iii) It is easy to see that if we write out the terms of total degree d in $P_{I,J}(n,m)$, this expression can be written as

$$\frac{1}{d!}\{e_0(I|J)n^d + \binom{d}{1} e_1(I|J)n^{d-1}m + \cdots + \binom{d}{d-1} e_{d-1}(I|J)nm^{d-1} + e_d(I|J)m^d\},$$

where $e_i(I|J) = e_{d-i,i}$, for $0 \leq i \leq d$.

(iv) Now suppose we fix $r, s \geq 1$. Then $e(I^r J^s)$ is determined by the lengths $\lambda(R/I^{rn}J^{sn})$ for $n >> 0$, which equal $P_{I,J}(rn, sn)$. If we substitute (rn, sn) into the equation in (iii), we get that the degree d term of $P_{I^r J^s}(n)$ is

$$\frac{1}{d!}\{e_0(I|J)r^r + \binom{d}{1} e_1(I|J)r^{d-1}s + \cdots + \binom{d}{d-1} e_{d-1}(I|J)rs^{d-1} + e_d(I|J)s^d\}n^d.$$

This shows that

$$e(I^r J^s) = e_0(I|J)r^d + \binom{d}{1} e_1(I|J)r^{d-1}s + \cdots + \binom{d}{d-1} e_{d-1}(I|J)rs^{d-1} + e_d(I|J)s^d.$$

Moreover, we also infer $e_i(I^r|J^s) = r^{d-i}s^i e_i(I|J)$, for all i. If we take $r = s = 1$ in the equation above, we have

$$e(IJ) = e_0(I|J) + \binom{d}{1} e_1(I|J) + \cdots + \binom{d}{d-1} e_{d-1}(I|J) + e_d(I|J). \qquad (\star)$$

(v) Let's see how the last formula in (iv) might lead to the Minkowski-type inequality for multiplicities discovered by Teissier. Consider three positive integers a, b, c.

If $a^{\frac{1}{d}} \leq b^{\frac{1}{d}} + c^{\frac{1}{d}}$, then raising this relation to the dth power, we get

$$a \leq \sum_{i=0}^{d} \binom{d}{i} b^{\frac{d-i}{d}} c^{\frac{i}{d}}.$$

It is clear that this last expression is equivalent to $a^{\frac{1}{d}} \leq b^{\frac{1}{d}} + c^{\frac{1}{d}}$.

If we take $a = e(IJ)$ and $b = e_0(I|J)$ and $c = e_d(I|J)$, then using $(\star)$, the Minkowski inequality for multiplicities holds if each

$$e_i(I|J) \leq e_0(I|J)^{\frac{d-i}{d}} \cdot e_d(I|J)^{\frac{i}{d}},$$

for $1 \leq i \leq d-1$.

Mixed Multiplicities

(vi) Set $e_i := e_i(I|J)$, for $0 \leq i \leq d$. Thus, Teissier's inequality holds if each $e_i^d \leq e_0^{d-i} e_d^i$.

Now suppose one could show $\frac{e_1}{e_0} \leq \frac{e_2}{e_1} \leq \cdots \leq \frac{e_d}{e_{d-1}}$. Then a relatively easy induction argument shows that $e_i^d \leq e_0^{d-i} e_d^i$, for $1 \leq i \leq d-1$.

For example, suppose the inequalities hold and $d = 3$, i.e., $\frac{e_1}{e_0} \leq \frac{e_2}{e_1} \leq \frac{e_3}{e_2}$. The first inequality gives $e_1^2 \leq e_0 e_2$. Multiplying this by e_1, we get $e_1^3 \leq e_0 e_1 e_2$. But we have $\frac{e_1}{e_0} \leq \frac{e_3}{e_2}$, so $e_1 e_2 \leq e_0 e_3$. Substituting this into the inequality $e_1^3 \leq e_0 e_1 e_2$ gives $e_1^3 \leq e_0^2 e_2$, as required.

Similarly, if we start with the inequality $\frac{e_2}{e_3} \leq \frac{e_3}{e_2}$, we have $e_2^2 \leq e_1 e_3$. Multiplying by e_2 gives $e_2^3 \leq e_1 e_2 e_3$. Since, by assumption, $e_1 e_2 \leq e_0 e_3$, substituting as before gives $e_3^2 \leq e_0 e_3^2$, which is what we want.

Thus, Tessier's theorem holds if one can show $\frac{e_1}{e_0} \leq \frac{e_2}{e_1} \leq \cdots \leq \frac{e_d}{e_{d-1}}$.

However, these inequalities hold, if $e_i^2 \leq e_{i-1} e_{i+1}$, for all $1 \leq i \leq d-1$.

(vii) Suppose $P(n,m)$ is a numerical polynomial in two variables of degree d and we write it in the form

$$P(n,m) = \sum_{i+j\le d} \binom{n+i}{i}\binom{m+j}{j}.$$

Then

$$P(n,m) - P(n-1,m) = \sum_{i+j\le d-1} e_{i,j}\binom{n+i-1}{i-1}\binom{m+j}{j}.$$

Now suppose R^* is a local ring of dimension $d-1$ having the property that

$$P_{I,J}(n,m) - P_{I,J}(n-1,m) = P_{IR^*,JR^*}(n,m).$$

It follows that $e_i(I^*|J^*) = e_i(I|J)$, for $0 \le i \le d-1$.

Similarly, if R' is a local ring of dimension $d-1$ having the property that

$$P_{I,J}(n,m) - P_{I,J}(n,m-1) = P_{IR',JR'}(n,m),$$

the same argument shows that $e_i(I'|J') = e_i(I|J)$, for $1 \le i \le d$.

Thus, if one can find rings R^* and R' satisfying these properties, one can prove the inequalities in (vii) by induction on the dimension of R.

Mixed Multiplicities

(viii) An element $a \in I$ is said to be superficial for I, J if there exists $c \geq 1$ such that

$$(I^n J^m : a) \cap I^c J^m = (I^{n-1} J^m : a),$$

for all $n > c$ and $m \geq 1$. We will see below that superficial elements exist, and that for $R^* := R/aR$,

$$P_{I,J}(n,m) - P_{I,J}(n-1,m) = P_{I^*,JR^*}(n,m).$$

This ultimately reduces the proof of Minkowski inequality for multiplicities to the two-dimensional case.

Mixed Multiplicities

(ix) Let L denote the stable value of $(0:\mathfrak{m}) \subseteq (0:\mathfrak{m}^2) \subseteq \cdots$. Then, we have seen that the image of $\mathfrak{m}$ in $S := R/L$ has positive grade.

Since I, J are $\mathfrak{m}$-primary, it follows that IS and JS also have positive grade.

Now, exactly the same proof as in item (ii) of the Applications of Superficial Elements from the previous section shows that $P_{I,J}(n,m)$ and $P_{IS,JS}(n,m)$ differ by a constant.

Since $\dim(R) = \dim(S) > 0$, this shows that $e_i(I|J) = e_i(IS, JS)$, for all i.

Thus, we are free to assume I and J have positive grade when working with mixed multiplicities.

(x) The discussion in the previous section concerning extending the residue field, in case k is finite, applies equally well in the current situation, so that the mixed multiplicities remain the same when we extend I and J to $R(x)$.

Thus, one may harmlessly assume that k is infinite.

Mixed Multiplicities

(xi) Let $K \subseteq I$ be a reduction of I, i.e., there exists n_0 such that $KI^{n_0} = I^{n_0+1}$. It follows that for all $n \geq n_0$, $K^{n-n_0}I^{n_0} = I^n$. Since $I^nJ^m \subseteq K^{n-n_0}J^m \subseteq I^{n-n_0}J^m$, for $n \geq n_0$, it follows that

$$P_{I,J}(n,m) \leq P_{K,J}(n-n_0,m) \leq P_{I,J}(n-n_0,m),$$

for $n \geq n_0$, and from this it follows that $e_i(I|J) = e_i(K|J)$, for all i (since the mixed multiplicities are positive).

The proof of Theorem F5 shows that when k is infinite, there exists an ideal $K \subseteq I$, generated by a system of parameters such that K is a reduction of I.

Thus, when k is infinite, we may replace I (or J) by a system of parameters and not change the mixed multiplicities.

(xii) $e_0(I|J) = e(I)$ and $e_d(I|J) = e(J)$. To see this, take n, m sufficiently large, so that the lengths $\lambda(R/I^nJ^m) = P_{I,J}(n,m)$. Now, fix $m = m_0$. Then $\lambda(R/I^nJ^{m_0}) =$

$$\frac{1}{d!}\{e_0(I|J)n^d + \binom{d}{1}e_1(I|J)n^{d-1}m_0 + \cdots + \binom{d}{d-1}e_{d-1}(I|J)nm_0^{d-1} + e_d(I|J)m_0^d\} + \cdots$$

This shows that the lengths $\lambda(R/I^nJ^{m_0})$ are given by a polynomial of degree d whose normalized leading coefficient is $e_0(I|J)$.

On the other hand, $\lambda(R/I^nJ^{m_0}) = \lambda(R/J^{m_0}) + \lambda(J^{m_0}/I^nJ^{m_0})$, for all n.

Therefore, the degree d polynomial giving the lengths of $J^{m_0}/I^nJ^{m_0}$, for n large, also has $e_0(I|J)$ as its normalized leading coefficient. In other words, $e_0(I|J) = e(I, J^{m_0})$, when we regard J^{m_0} as an R-module.

Since $\dim(R/J^{m_0}) = 0$, additivity of the multiplicity symbol $e(I, -)$ applied to the exact sequence

$$0 \to J^{m_0} \to R \to R/J^{m_0} \to 0,$$

shows that $e(I, J^{m_0}) = e(I, R) = e(I)$. Thus, $e_0(I|J) = e(I)$.The proof that $e_d(I|J) = e(J)$ is similar.

Two-variable Hilbert-Samuel Polynomials

Our first goal is to show the existence of $P_{I,J}(n,m)$ while at the same time showing that for all $0 \leq i \leq d$, $e_i(I|J) > 0$. For this, we need superficial elements relative to a pair of ideals. We also need a bigraded version of the Artin-Rees Lemma. We just indicate the proof, because it is essentially the same as in the usual case:

Two variable version of Artin-Rees. Suppose $I, J, K \subseteq R$ are ideals. Then there exists $u, v \geq 1$ such that for all $n \geq u$ and $m \geq v$, $I^n J^m \cap K \subseteq I^{n-u} J^{m-v} K$. To see this, One uses the bigraded Rees algebra $\mathcal{R} := R[It, Js]$ and considers the ideal $\mathcal{K} = \bigoplus (K \cap I^n J^m) t^n s^m \subseteq \mathcal{R}$.

This is a homogenous ideal with respect to the bigrading on $\mathcal{R}$, so it has a set of homogeneous generators. Take u greater than any exponent of t among these generators and v greater than the exponent of s among these generators.Then for any element $b \in I^n J^m \cap K$ with $n \geq u$ and $m \geq v$, $bt^n s^m \in \mathcal{K}$.

Express this element in terms of the generators of $\mathcal{K}$, and read off the homogeneous coefficients of the generators, to get the desired conclusion.

Two-variable Hilbert-Samuel Polynomials

Proposition B6. Let $(R, \mathfrak{m}, k)$ be a local ring with infinite residue field and positive depth. Let $I, J \subseteq R$ be $\mathfrak{m}$-primary ideals.

Then there exist $c > 0$ and a non-zerodivisor $a \in I$, that is also a minimal generator of I, such that $(I^n J^m : a) = I^{n-1} J^m$, for all $n > c$ and all $m \geq 1$.

Proof. We use the Rees ring $\mathcal{R} := R[It, Js] = \bigoplus_{n,m \geq 0} I^n J^m t^n s^m$, where t, s are indeterminates over R. This is a bi-graded algebra, generated in degrees (1,0) and (0,1) over R.

Let $Q_1, \ldots, Q_r$ be the associated primes of $I\mathcal{R}$ not containing It and $Q_{r+1} \ldots, Q_h$ be the remaining associated primes of $I\mathcal{R}$.

Choose $c_0 > 0$ such that $(It)^{c_0}$ is contained in the Q_i-primary component of $I\mathcal{R}$, for $r + 1 \leq i \leq h$.

We set $J_i := \{r \in R \,|rt \in Q_i\}$, for $r + 1 \leq i \leq h$. Then as before, $I \not\subseteq J_i$, so that $I \cap J_i$ is properly contained in I.

Two-variable Hilbert-Samuel Polynomials

Let $P_1, \ldots, P_b$ be the associated primes of R, and note that $W_i := I \cap P_i$ is properly contained in I.

Finally, take $J \subseteq I$ so that $(J + \mathfrak{m}I)/\mathfrak{m}I \subseteq I/\mathfrak{m}I$ has dimension one less that the dimension of the vector space $I/\mathfrak{m}I$.

Then the subspaces

$$(J_i + \mathfrak{m}I)/\mathfrak{m}I, (W_i + \mathfrak{m}I)/\mathfrak{m}I, (J + \mathfrak{m}I)/\mathfrak{m}I$$

are all proper subspaces of $I/\mathfrak{m}I$, so there exists $a \in I$ whose image in $I/\mathfrak{m}I$ avoids these subspaces.

Thus a is a minimal generator of I and also a non-zerodivisor.

Two-variable Hilbert-Samuel Polynomials

Now suppose $n > c_0$ and $r \in (I^n J^m : a) \cap I^{c_0} J^m$. If $n = c_0 + 1$, then $c_0 = n - 1$, so $r \in I^{n-1} J^m$, which is what we want.

Suppose $n > c_0 + 1$. Then $rt^{c_0} s^m \in \mathcal{R}$ and $rt^{c_0} s^m \cdot at \in I^n J^m t^{c_0+1} s^m \in I\mathcal{R}$. By definition of c_0, $rt^{c_0} s^m$ belongs to every Q_i-primary component of $I\mathcal{R}$, with $r + 1 \leq i \leq h$.

On the other hand, the choice of a forces $rt^{c_0} s^m$ to be in the Q_j-primary components of $I\mathcal{R}$, for $1 \leq i \leq r$.

Thus, $rt^{c_0} s^m \in I\mathcal{R}$. This implies $r \in I^{c_0+1} J^m$.

We may repeat the argument until we arrive at $r \in I^{n-1} J^m$.

Two-variable Hilbert-Samuel Polynomials

Finally, let u, v be chosen so that $(a) \cap I^n J^m \subseteq aI^{n-u} J^{m-v}$, for $n \geq u$ and $m \geq v$.

Take p such that $I^p \subseteq J$, so that $I^{pv} \subseteq J^v$.

Now suppose $n > pv + u + c_0$ and take $r \in (I^n J^m : a)$. Then

$$ra \in I^n J^m \subseteq I^{u+c_0} J^{m+v},$$

so $ra \in (a) \cap I^{u+c_0} J^{m+v}$ and we can write $ra = ax$, for $x \in I^{c_0} J^m$.

Thus, $r = x \in I^{c_0} J^m$, since a is a non-zerodivisor.

Therefore, by the previous paragraph, $r \in (I^n J^m : a) \cap I^{c_0} J^m = I^{n-1} J^m$.

Taking $c := pv + u + c_0$ shows that $(I^n J^m : a) = I^{n-1} J^m$, for all $n > c$, and all $m \geq 1$. $\square$