April 27: Integral closure of ideals, multiplicity, and the Rees multiplicity theorem, part 4

Theorem M5(Associativity formula). Let $(R, \mathfrak{m}, k)$ be a local ring, $I \subseteq R$ an $\mathfrak{m}$-primary ideal and M a finitely generated R-module. Then

$$e(I, M) = \sum_P e(I, R/P)\lambda(M_P),$$

where the sum is taken over all primes $P \in \mathrm{Spec}(R)$ such that $\dim(R/P) = \dim(R)$. In particular,

$$e(I) = \sum_P e((IR + P)/P)\lambda(R_P).$$

where the sum is taken over all primes $P \in \mathrm{Spec}(R)$ such that $\dim(R/P) = \dim(R)$.

Proof. Let $(0) = M_0 \subseteq M_1 \subseteq M_r = M$ be a filtration of M with each $M_i/M_{i-1} \cong R/Q_i$, for each $1 \leq i \leq r$, and $Q_i \in \mathrm{Spec}(R)$.

Now, for each i, there is a short exact sequence

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0,$$

from which it follows that $e(I, M_i) = e(I, M_{i-1}) + e(I, M_i/M_{i-1})$.

Multiplicities

Putting these equations all together shows that $e(I, M) = \sum_i e(I, R/Q_i)$.

By Corollary L5, $e(I, R/Q_i) \neq 0$ if and only if $\dim(R/Q_i) = \dim(R)$.

Thus, the only terms in the sum $e(I, M) = \sum_i e(I, R/Q_i)$ that are non-zero, are the terms for which $\dim(R/Q_i) = \dim(R)$.
Suppose Q_i satisfies $\dim(R/Q_i) = \dim(R)$. Q_i may appear more than once. We note that the number of times R/Q_i appears in the filtration of M is $\lambda(M_{Q_i})$.

Localize R and M at Q_i. Then M_{Q_i} is a finite length R_{Q_i}-module, and the original filtration localizes to a new filtration whose factors are just $k(Q_i)$, and none of the original factors corresponding to R/Q_i are lost.

This new filtration is a composition series for M_{Q_i} and the number of factors is $\lambda(M_{Q_i})$, which gives what we want.

Multiplicities

Finally, if $P \in \mathrm{Spec}(R)$ satisfies $\dim(R/P) = \dim(R)$ and P does not appear in the filtration of M, then $M_P = 0$. So it can harmlessly be included in the sum

$$e(I, M) = \sum_i e(I, R/Q_i).$$

This shows $e(I, M) = \sum_P e(I, R/P)\lambda(M_P)$, where the sum is taken over all primes $P \in \mathrm{Spec}(R)$ such that $\dim(R/P) = \dim(R)$. $\square$

Multiplicities

How do we use the associativity formula in the proof of the Rees multiplicity theorem?

One first reduces to the case R is complete.

From $e(I) = e(J)$, by the associativity formula we have

$$\sum_P e(I, R/P)\lambda(R_P) = \sum_P e(J, R/P)\lambda(R_P),$$

where the sum is taken over all primes $P \in \mathrm{Spec}(R)$ such that $\dim(R/P) = \dim(R)$.

Since each term $e(I, R/P) \leq e(J, R/)$, each of these terms must be equal.

Thus the problem is reduced to a complete local domain.

Note that if I and J have the same integral closure modulo each minimal prime, they have the same integral closure.

Since the original R was quasi-unmixed, the primes $P \in \mathrm{Spec}(R)$ such that $\dim(R/P) = \dim(R)$ are the minimal primes of R.

Observation. Suppose $R \subseteq S$ are integral domains with quotient fields $K \subseteq L$, respectively. Assume S is finite over R, so that L is a finite extension of K.

Then there exists a free R-module $F \subseteq S$ and a non-zero element $r \in R$ such that $rS \subseteq F$. To see this, let $U \subseteq R$ be the set of non-zero elements. Then S_U is finite, and hence integral, over $R_U = K$.

Thus, S_U is a field. Since $S_U \subseteq L_U = L$, $S_U = L$.

Thus, every element in the quotient field of S is a fraction of the form $\frac{s}{u}$, where $s \in S$ and $u \in R$.

Now suppose $\frac{s_1}{u_1}, \dots, \frac{s_t}{u_t}$ form a basis for L over K. Then the R-module $F := Rs_1 + \cdots + Rs_t$ is a free R-module of rank t contained in S.

On the other hand, if $s \in S$, we can write

$$s = \alpha_1 \frac{s_1}{u_1} + \cdots + \alpha_t \frac{s_t}{u_t},$$

with each $\alpha_i \in K$. Clearly denominators shows that there exists $0 \neq r_0 \in R$ such that $r_0 s \in F$.

Since S is a finite R-module, there exists $0 \neq r \in R$ with $rS \subseteq F$.

Proposition N5. Let $(R, \mathfrak{m}, k)$ be a local domain with quotient field K and S an integral domain that is a finite R-module. Let L denote the quotient field of S, $\mathfrak{n}_1, \ldots, \mathfrak{n}_r$ denote the maximal ideals of S, and set $k_i := S/\mathfrak{n}_i$. Then,

$$e(I) \cdot [L : K] = \sum_{i=1}^{r} e(IS_{\mathfrak{n}_i})[k_i : k].$$

Proof. Let $F \subseteq S$ be as in the observation above and take $0 \neq r \in R$ such that $rS \subseteq F$. Then r annihilates S/F, and thus $\dim(S/F) < \dim(R)$.

Therefore, $e(I, F/S) = 0$. Additivity of the multiplicity symbol applied to the exact sequence

$$0 \to F \to S \to S/F \to 0,$$

gives $e(I, S) = e(I, F)$. Here we are thinking of S and F as R-modules.

Since F is free of rank $[L : K]$ over R, we have

$$e(IF) = e(I)[L : K].$$

We must now show that $e(I, S) = \sum_{i=1}^{r} e(IS_{\mathfrak{n}_i})[k_i : k]$.

Multiplicities

On the one hand,

$$e(I, S) = \lim_{n\to\infty} \frac{n^d}{d!}\lambda_R(S/I^nS).$$

On the other hand, since $\lambda_R(k_i) = \lambda_k(k_i) = [k_i : k]$, if H is an $S_{\mathfrak{n}_i}$-module with finite length, additivity of the length function shows that

$$\lambda_R(H) = [k_i : k]\lambda_{S_{\mathfrak{n}_i}}(H).$$

Thus,

$$\lambda_R(S_{\mathfrak{n}_i}/I^nS_{\mathfrak{n}_i}) = [k_i : k]\lambda_{S_{\mathfrak{n}_i}}(S_{\mathfrak{n}_i}/IS_{\mathfrak{n}_i}).$$

Since $I^nS = (I^nS_{\mathfrak{n}_1} \cap S) \cap \cdots \cap (I^nS_{\mathfrak{n}_r} \cap S)$, and the ideals $I^nS_{\mathfrak{n}_i} \cap S$ are co-maximal, we have an isomorphism of R-modules

$$S/I^nS \cong S/(I^nS_{\mathfrak{n}_1} \cap S) \oplus \cdots \oplus S/(I^nS_{\mathfrak{n}_r} \cap S),$$

for all $n \geq 1$. Thus,

$$\lambda_R(S/I^nS) = \lambda_R(S/(I^nS_{\mathfrak{n}_1} \cap S)) + \cdots + \lambda_R(S/(I^nS_{\mathfrak{n}_r} \cap S)).$$

However, $\mathfrak{n}_i$ is the only maximal ideal of S containing $I^nS_{\mathfrak{n}_i} \cap S$, so that the ring $S/(I^nS_{\mathfrak{n}_i} \cap S)$ is local, i.e., $S/(I^nS_{\mathfrak{n}_i} \cap S) = S_{\mathfrak{n}_i}/I^nS_{n_i}$, for all i. Therefore,

$$\lambda_R(S/I^nS) = \lambda_R(S_{\mathfrak{n}_1}/I^nS_{\mathfrak{n}_1}) + \cdots + \lambda_R(S_{\mathfrak{n}_r}/I^nS_{\mathfrak{n}_r}).$$

Multiplicities

From the first sentence of this paragraph we have

$$\lambda_R(S/I^nS) = [k_1 : k]\lambda_{S_{\mathfrak{n}_1}}(S_{\mathfrak{n}_1}/I^nS_{\mathfrak{n}_1}) + \cdots + [k_r : k]\lambda_{S_{\mathfrak{n}_r}}(S_{\mathfrak{n}_r}/I^nS_{\mathfrak{n}_r}).$$

multiplying this last equation by $\frac{n^d}{d!}$ and taking the limit as $n \to \infty$ gives

$$e(IS) = \sum_{i=1}^{r} e(IS_{\mathfrak{n}_i})[k_i : k]$$

which is what we want. $\square$

The following theorem due to DK provides a natural way to connect the multiplicity of an I-primary ideal to its Rees valuations rings.

Theorem O5. Let $(R, \mathfrak{m}, k)$ be a local domain of dimension at least two, and $I = (a_1, \ldots, a_d)R$ an ideal generated by a system of parameters. Set $T := R[\frac{a_1}{a_d}, \cdots, \frac{a_{d-1}}{a_d}]_{\mathfrak{m}R[\frac{a_1}{a_d}, \cdots, \frac{a_{d-1}}{a_d}]}$. Then $e(I) = e(IT)$.

Proof. Without loss of generality, we may assume that k is infinite. We induct on $\dim(R)$. Suppose $\dim(R) = 2$.

It follows from Proposition J5, that there exists $a_1' \in I$ such that a_1' is a minimal generator and a superficial element for I. Moreover, a_1' can be chosen to have the form $a_1 + ra_2$, for some $r \in R$. Note that $I = (a_1', a_2)$ and $R[\frac{a_1'}{a_2}] = R[\frac{a_1}{a_2}]$, so we may begin again assuming that a_1 is a superficial element for I.

We consider the natural homomorphism ϕ from the polynomial ring $R[x] \to R[\frac{a_1}{a_2}]$ taking x to $\frac{a_1}{a_2}$. As in the proof of Proposition D5, we let K be the kernel of this homomorphism, and $L := g(x)R[x]$, where $g = a_2 x - a_1$.

Multiplicities

We still have that $a_2^c \cdot K \subseteq gR[x]$, for some c. Note that K_S is the kernel of ϕ_S, the map obtained by inverting the elements S in $R[x]$, not in $\mathfrak{m}R[x]$. We will write $R(x)$ for $R[x]_S$ and note that ϕ_S maps $R(x)$ onto T.

We claim that g is superficial for $IR[x] = (g, a_2)R[x]$. To see this, suppose $c \geq 1$ satisfies $(I^n : a_1) = I^{n-1}$, for $n \geq c$. (Note that x_1 is a non-zerodivisor.) Suppose $f \cdot g \in I^n R[x]$, where $f = \sum_{i=0}^{s} r_i x^i$.

$$fg = -a_1 r_0 + (a_2 r_0 - a_1 r_1)x + \cdots + (a_2 r_{s-1} - a_1 r_s)x^s + a_2 r_2 x^{s+1}.$$

It follows that $r_0 \in (I^n : a_1)$, so $r_0 \in I^{n-1}$. Therefore, $a_2 r_0 \in I^n$, which implies, $r_1 \in (I^n : a_1)$. Thus, $r_1 \in I^{n-1}$.

Inductively, we see that $r_i \in I^{n-1}$ for all i, so $f \in I^{n-1}R[x]$.

Therefore, g superficial for $IR(x)$.

Thus $e(I) = e(IR(x)) = e(IR(x)/gR(x))$.

Multiplicities

Set $A := R(x)/gR(x))$, so $e(I) = e(IA)$. Over A we have an exact sequence of A-modules

$$0 \to KA \to A \to T \to 0.$$

Note that a power of a_2 annihilates KA, and since a_2 is part of a system of parameters for A (in this case, an entire system of parameters),this means $\dim(KA) < \dim(A)$.

Therefore, $e(IA, KA) = 0$. By the additivity of the multiplicity symbol, $e(IA, A) = e(I, T)$, which gives what we want.

Now suppose the result holds for local domains of dimension $d-1$. As before, we may assume a_1 is superficial for I. Set $T_1 := R[\frac{a_1}{a_d}]_{\mathfrak{m}R[\frac{a_1}{a_d}]}$.

Exactly the same proof as above shows that $e(I) = e(IT_1)$. The ring T_1 is a $(d-1)$-dimensional local ring with system of parameters $a_2, \ldots, a_d$. Thus, by induction, $e(IT_1) = e(IT^*)$, where $T^* = T_1[\frac{a_2}{a_d}, \ldots, \frac{a_{d-1}}{a_d}]$.

However, it is readily seen (as in the proof of Proposition E5) that

$$T^* = R[\frac{a_1}{a_d}, \cdots, \frac{a_{d-1}}{a_d}]_{\mathfrak{m}R[\frac{a_1}{a_d}, \cdots, \frac{a_{d-1}}{a_d}]} = T,$$

which completes the proof. □

The next two theorems are due to D. Rees. The proofs are due to DK. The proofs use the following standard fact, namely, that if I is an $\mathfrak{m}$-primary ideal in a local ring with infinite residue field, there exists an ideal $J \subseteq R$, generated by a system of parameters, such that $e(J) = e(I)$.

This is essentially equivalent to the conclusion of Proposition F5, and is the form of Proposition F5 first given by Northcott and Rees.

To see this, note that the proof of Proposition F5 shows that there exists an ideal $J \subseteq I$ generated by a system of parameters such that $JI^n = I^{n+1}$ for all n large, say $n \geq n_0$. Therefore, $J^{n-n_0} I^{n_0} = I^n$, for all $n > n_0$. It follows that for all $n > n_0$, $I^n \subseteq J^{n-n_0} \subseteq I^{n-n_0}$. Thus, for $n >> 0$,

$$P_I(n - n_0) \leq P_J(n) \leq P_I(n),$$

which shows $e(J) = e(I)$.

Multiplicities

In a similar vein, suppose J is an $\mathfrak{m}$-primary ideal and $a \in R$ is integral over J. Then there exists an equation

$$a^n + j_1 a^{n-1} + \cdots + j_n = 0,$$

with each $j_s \in J^s$. This implies $a^n \in J(x, J)^{n-1}$, which gives $(a, J)^n = J(a, J)^{n-1}$.

Thus, the same argument as above shows $e(a, J) = e(J)$. Since J is finitely generated, this shows $e(J) = e(\overline{J})$.

Therefore, if $\overline{I} = \overline{J}$, for $\mathfrak{m}$-primary ideals $I, J \subseteq R$, then $e(I) = e(J)$.

Multiplicities

Remark. Suppose V is a DVR with uniformizing parameter π and quotient field K. Then any non-zero element $a \in K$ can be written uniquely as $u\pi^n$, for some $n \in \mathbb{Z}$.

This enables one to define a function $v : K \to \mathbb{Z} \cup \infty$ by $v(a) = n$, if $a \in K$ is non-zero and $a = u\pi^n$, and $v(0) = \infty$.

The function v is called the *valuation associated to V*. If $J \subseteq V$ is an ideal, we write $v(J)$ for $v(a)$, where $J = aV$. This is the minimum value $v(j)$, with $j \in J$.

Note that if $v(J) = e$, then $\lambda_V(V/J^n) = en$ for all n, so that $e(J) = e$, i.e., $e(J) = v(J)$, for all ideals $J \subseteq V$.

Theorem Q5. Let $(R, \mathfrak{m}, k)$ be an analytically unramified local domain with infinite residue field and $I \subseteq R$ an $\mathfrak{m}$-primary ideal. Then there exist finitely many DVRs $V_1, \ldots, V_r$ between R and its quotient field, and finitely may positive integers $d_1, \ldots, d_r$ such that

$$e(I) = \sum_{i=1}^{r} d_i v_i(I),$$

where v_i is the valuation associated to V_i.

Proof. If we take $J \subseteq R$ such that J is generated by a system of parameters with $e(J) = e(I)$, we may replace J by I and begin again, assuming that $I = (a_1, \ldots, a_d)R$ is generated by a system of parameters.

Taking T as in Theorem O5, we have $e(I) = e(IT)$.

By Rees's theorem on analytically unramified local domains, T' is a finite T-module. Since T and T' have the same quotient field, applying Proposition N5 gives

$$e(IT) = \sum_{i=1}^{r} d_i e(IV_i),$$

where $d_i = [V_i/\mathfrak{m}_{V_i} : k]$. By the preceding remark, $e(IV_i) = v_i(I)$, which completes the proof. $\square$