April 24: Integral closure of ideals, multiplicity, and the Rees multiplicity theorem, part 3

Hilbert Polynomials

Therem H5. Let $A = \bigoplus_{n\geq 0} A_n$ be a finitely generated R-algebra, where $A_0 = (R, \mathfrak{m}, k)$ is a local Artinian ring. We assume that A is a standard, graded R-algebra, i.e., $A = R[A_1]$. Let $M = \bigoplus_{n\geq 0} M_n$ be a finitely generated, graded A-module. Then $H_M(n) := \lambda_R(M_n) < \infty$, for all n and $H_M(n)$ agrees with a numerical polynomial $P_M(x)$ of degree $\dim(M) - 1$, for $n >> 0$.

Definition. The function $H_M(n)$ above is called the Hilbert function of M, while the polynomial $P_M(x)$ is called the Hilbert polynomial of M.

We now want to apply the theorem above to the associated graded ring of an $\mathfrak{m}$-primary ideal. Recall that the associated graded ring of a Noetherian ring R with respect to an ideal $I \subseteq R$ is the ring,

$$\mathcal{G} := \bigoplus_{\mathfrak{n} \geq 0} I^n/I^{n+1} = \mathcal{R}/I\mathcal{R},$$

where $\mathcal{R}$ is the Rees ring of R with respect to I.

Note that as an R/I-algebra, $\mathcal{G} = R/I[I/I^2]$, so that $\mathcal{G}$ is a standard graded, finitely generated R/I-algebra. If I is $\mathfrak{m}$-primary, then $R/I = \mathcal{G}_0$ is an Artinian ring, so Theorem H5 applies.

We give two versions of the Hilbert polynomial associated to $\mathfrak{m}$-primary ideal I.

Corollary I5. Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d and $I \subseteq R$ an $\mathfrak{m}$-primary ideal.

(i) The function $\tilde{H}_I(n) := \lambda_R(I^n/I^{n+1})$ agrees with a numerical polynomial $\tilde{P}_I(x)$ of degree $d-1$, for $n >> 0$.

(ii) The function $H_I(n) := \lambda(R/I^{n+1})$ agrees with a numerical polynomial $P_I(x)$ of degree d, for $n >> 0$.

Multiplicities

Definition. Maintaining the notation from Corollary I5, it follows from the discussion above on numerical polynomials, that we can write

$$P_I(x) = e_0\binom{x+d}{d} + e_1\binom{x+d-1}{d-1} + \cdots + e_{d-1},$$

where each $e_j \in \mathbb{Z}$ and $e_0 > 0$. The integer e_0 is called the multiplicity of I and is denoted $e(I)$.

Note that if we write $P_I(x)$ in the form $q_0x^d + q_1x^{d-1} + \cdots + q_d$, with each $q_j \in \mathbb{Q}$, then $q_0 = \frac{e(I)}{d!}$. Thus,

$$e(I) = \lim_{n\to\infty} \frac{d!}{n^d} \cdot \lambda(R/I^{n+1}).$$

Note that $e(I)$ is also the normalized leading coefficient of $\tilde{P}_I(x)$, since $P_I(x) - P_I(x-1) = \tilde{P}_I(x)$.

Superficial elements

In the proof of Theorem F5 and Theorem H5 we used the fact that the associated graded ring of an ideal has the same dimension as the ring. For some of our results below, we will need to refine the superficial element argument from above.

Definition. Let $I \subseteq R$ be a ideal in the Noetherian ring R. $a \in I$ is said to be a superficial element for I if there exists $c > 0$ such that $(I^n : a) \cap I^c = I^{n-1}$, for $n \geq c$.

Proposition J5. Let $(R, \mathfrak{m}, k)$ be a local ring with infinite residue field and suppose $I \subseteq R$ is an ideal having height greater than zero. Then:

(i) There exists $a \in I$, a superficial element for I, that is also a minimal generator for I.

(ii) If $\mathrm{grade}(I) > 0$, then there exists a superficial element for I that is both a minimal generator for I and a non-zerodivisor.

(iii) If $a \in I$ is superficial for I, and a non-zerodivisor, then $(I^n : a) = I^{n-1}$, for $n >> 0$.

Superficial elements

Proof. We will prove parts (i) and (ii) at the same time. Let $\mathcal{G}$ denote the associated graded ring of R.By the proof of Lemma G5, if $a \in I/I^2$ has the property that its image $\overline{a}$ in $\mathcal{G}_1$ does not belong to any associated prime of (0) in $\mathcal{G}$ that contains $\mathcal{G}_+$, then, in the ring $\mathcal{G}$, $\overline{a}$ is superficial in the sense described there.

Suppose $\overline{a}$ is such an element (which exists, by Lemma G5). Let c be as in Lemma G5. Then suppose $b \in (I^n : a) \cap I^c$, with $n \geq c$.

If $b \notin I^{n-1}$, choose e maximal such that $e \geq c$, yet $e < n - 1$ with $b \in I^e$. Then $\overline{b} \in \mathcal{G}_e$.

Now, on the one hand, $\overline{a} \in \mathcal{G}_1$, so $\overline{a} \cdot \overline{b} \in \mathcal{G}_{e+1}$. But $ab \in I^n$ and $n > e + 1$, so $\overline{a} \cdot \overline{b} = 0$ in $\mathcal{G}$. Since $e \geq c$, this means $\overline{b} = 0$, i.e., $b \in I^{e+1}$, contrary to the choice of e.

Thus, in fact, $b \in I^{n-1}$, so a is a superficial element for I.

Superficial elements

Now let us write $\mathcal{G} = \mathcal{R}/I\mathcal{R}$, where $\mathcal{R}$ is the Rees ring of R with respect to I. Then a primary decomposition of (0) in $\mathcal{G}$ corresponds to a primary decomposition of $I\mathcal{R}$.

Let $Q_1, \ldots, Q_r$ be the associated primes in a primary decomposition of $I\mathcal{R}$ that do not contain $\mathcal{R}_+$. Let $J_i = \{a \in R \mid at \notin Q_i\}$.

Note that by definition, $I \not\subseteq J_i$, therefore $J_i \cap I$ is properly contained in I.

Write $d := \dim_k(I/\mathfrak{m}I)$ and take $J = (a_2, \ldots, a_d)R$, where the images of the a_i in $I/\mathfrak{m}I$ are linearly independent.

Then the subspaces $(J_1 + \mathfrak{m}I)/\mathfrak{m}I, \ldots, (J_r + \mathfrak{m}I)/\mathfrak{m}I, (J + \mathfrak{m}I)/\mathfrak{m}I$ are proper subspaces of the k-vector space $I/\mathfrak{m}I$.

Take $a \in I$ such that its image in $I/\mathfrak{m}I$ does not belong to any of these subspaces.

Superficial elements

Then, one the one hand, $a \notin J_i$ all i, so $at \notin Q_i$ all i. Thus, by our discussion above, a is a superficial element for I.

On the other hand, since the image of a in $I/\mathfrak{m}I$ does not belong to $(J + \mathfrak{m}I)/\mathfrak{m}I$, the images of $a, a_2, \ldots, a_d$ in $I/\mathfrak{m}I$ are linearly independent over k, and thus form a minimal set of generators of I.

In particular, a is a minimal generator of I. This gives (i).

If in addition $\mathrm{grade}(I) > 0$, let $P_1, \ldots, P_s$ denote the associated primes of R and set $W_i := P_i \cap I$, for each i. Then since $I \not\subseteq P_i$, W_i is properly contained in I.

Thus the subspaces $(W_i + \mathfrak{m}I)/\mathfrak{m}I$ are proper subspaces of $I/\mathfrak{m}I$ and if a is chosen so that its image in $I/\mathfrak{m}I$ also avoids these subspaces, then $a \notin P_i$, for all i, and thus, we have that a is also a non-zerodivisor.

Superficial elements

Finally, take $a \in I$ as in the statement of (iii). By the Artin-Rees lemma, there exists $k > 0$ such that

$$I^n \cap (a) \subseteq I^{n-k}a,$$

for all $n \geq k$.

Let c be as in (i). For any $n \geq c + k$, suppose $ra \in I^n$.

Then $ra \in I^n \cap (a) \subseteq I^{n-k}a$. We can write $ra = ia$, with $i \in I^{n-k}$.

Then $(r - i)a = 0$, and thus, $r = i$, since a is a non-zerodivisor.

Therefore, $r \in I^{n-k} \subseteq I^c$, since $n \geq k + c$.

Therefore $r \in (I^n : a) \cap I^c = I^{n-1}$, which is what we want. □

Superficial elements

Applications of Superficial Elements. We assume that $(R, \mathfrak{m}, k)$ is a local ring of dimension $d > 0$, and $I \subseteq R$ is an $\mathfrak{m}$-primary ideal.

(i) Suppose $a \in I$ is a superficial element and a non-zero divisor. Set $R^* = R/(a)$.

Then $e(I) = e(IR^*)$. To see this, note that by (iii) in the proposition above, $(I^n : a) = I^{n-1}$, for n sufficiently large. Thus, the sequence

$$0 \to R/I^{n-1} \stackrel{\cdot a}{\to} R/I^n \to R/(I^n, a)R \to 0,$$

is exact for large n. Since $R/(I^n, a)R = R^*/I^n R^*$, we have

$$P_I(n) - P_I(n-1) = P_{IR^*}(n).$$

for $n >> 0$. Thus, $P_{IR^*}(x) = P_I(x) - P_I(x-1)$.

If $f(x)$ is a polynomial of degree d, then $f(x) - f(x-1)$ is a polynomial of degree $d-1$ whose leading coefficient is d times the leading coefficient of $f(x)$. Thus, the normalized leading coefficient of $P_{IR^*}(n)$ is $e(I)$, which gives $e(IR^*) = e(I)$.

Superficial elements

(ii) This item shows that one can often assume that the ideal I has a superficial element that is a non-zerodivisor.

Suppose $\mathrm{depth}(R) = 0$, so that I does not contain a non-zerodivisor. Let L be the stable value of the increasing chain of ideals $(0 : I) \subseteq (0 : I^2) \subseteq \cdots$. Let's first note that $I^n \cap L = 0$, for $n >> 0$.

Suppose $L = (0 : I^c)$. By the Artin-Rees lemma, there exists k such that $I^n \cap L \subseteq I^{n-k}L$. When $n - k \geq c$, $I^{n-1}L = 0$, which gives what we want.

Now, for all n, we have an exact sequence

$$0 \to (I^n + L)/I^n \to R/I^n \to \tilde{R}/I^n\tilde{R} \to 0.$$

Thus $\lambda(R/I^n) = \lambda(\tilde{R}/I^n\tilde{R}) + \lambda((I^n + L)/I^n)$. However

$$(I^n + L)/I^n) \cong L/(I^n \cap L) = L.$$

for $n >> 0$.

Thus, $P_I(x) = P_{I\tilde{R}}(x) + \lambda(L)$.

Superficial elements

Since $\dim(R) > 0$, $P_I(x)$ has degree greater than zero. Thus, the normalized leading coefficients of $P_I(x)$ and $P_{I\tilde{R}}(x)$ are the same, so that $e(I) = e(I\tilde{R})$.

But now, $I\tilde{R}$ has grade at least one. To see this, suppose $\mathrm{grade}(I\tilde{R}) = 0$.

Then there exists $0 \neq \tilde{r} \in \tilde{R}$ such that $\tilde{r} \cdot I\tilde{R} = 0$. Interpreting this in R, we have $rI \subseteq L$. Thus, $rI \cdot I^c = 0$.

Therefore, $r \in (0 : I^{c+1}) = (0 : I^c) = L$, a contradiction. Therefore, $\mathrm{grade}(I\tilde{R}) > 0$.

This shows that we can always pass to a ring in which the multiplicity of I stays the same, but the image of I has a superficial element that is a non-zerodivisor (when the residue field is infinite).

Superficial elements

(iii) Assume I is generated by a system of parameters. Then $e(I) \leq \lambda(R/I)$. To see this, we may assume k is infinite.

Now induct on d. Suppose $d = 1$, so $I = aR$. Let L be as in (ii). Then $e(aR) = e(a\tilde{R})$. Now in $\tilde{R}$, the image of a is a non-zerodivisor, so we we have that

$$\tilde{R}/(\tilde{a}) \cong a^{n-1}\tilde{R}/a^n\tilde{R},$$

for all n.

Applying this to the filtration

$$(0) \subseteq a^n\tilde{R} \subseteq a^{n-1}\tilde{R} \subseteq \cdots \subseteq \tilde{R},$$

shows that $\lambda(\tilde{R}/a^n\tilde{R}) = \lambda(\tilde{R}/a\tilde{R}) \cdot n$, for all n.

Thus,

$$\lambda(\tilde{R}/a\tilde{R}) = e(a\tilde{R}) = e(aR).$$

Since $\lambda(\tilde{R}/a\tilde{R}) \leq \lambda(R/aR)$, we have $e(aR) \leq \lambda(R/aR)$.

Superficial elements

The inductive step is similar. Let L be as in (ii) and $\tilde{R} = R/L$. Then $e(I) = e(I\tilde{R})$ and $\lambda(\tilde{R}/I\tilde{R}) \leq \lambda(R/I)$.

Thus, if we can prove the inequality we seek over $\tilde{R}$, it will hold in R.

Note, that L is a nilpotent ideal, so that $\dim(\tilde{R}) = \dim(R)$, and hence $I\tilde{R}$ is generated by a system of parameters.

Changing notation, we now assume that $\mathrm{grade}(I) > 0$. Now, by Proposition J5, we may assume that the first generator, say a, of I is a superficial element and a non-zerodivisor.

Setting $R^* := R/aR$, by (ii), we have $e(IR^*) = e(I)$. IR^* is generated by a system of parameters, so by induction

$$e(IR^*) \leq \lambda(R^*/IR^*).$$

But $R/I \cong R^*/IR^*$, so $\lambda(R/I) = \lambda(R^*/IR^*)$, which completes the proof.

Remark. Since our main goal is the Rees multiplicity theorem, we are interested in the relevant results concerning the multiplicity of ideals in local rings. However, certain technical results needed along the way are made easier by extending the notion of multiplicity to modules.

To that end, let $(R, \mathfrak{m})$ be a local ring of dimension d, $I \subseteq R$ an $\mathfrak{m}$-primary ideal, and M a finitely generated R-module. Then the module

$$\mathcal{M} := \bigoplus_{n \geq 0} I^n M / I^{n+1} M$$

is a finitely generated $\mathcal{G}$-module.

Thus, by Theorem H5, the lengths $\lambda_R(I^n/I^{n+1}M)$ agree with a numerical polynomial of degree $\dim(\mathcal{M}) - 1$, for $n >> 0$.

In particular, this polynomial has degree less than or equal to $d - 1$. It follows that the polynomial $P_{I,M}(x)$ that agrees with $\lambda(M/I^nM)$ for $n >> 0$ has degree less than or equal to d.

This enables us to define a multiplicity symbol $e(I, -)$ as follows:

$$e(I, M) = \lim_{n \to \infty} \frac{d!}{n^d} \cdot \lambda(M/I^nM).$$

Of course, $e(I, R) = e(I)$.

Multiplicities

The next proposition plays a key role in the proof of the associativity formula for multiplicities.

The associativity formula often allows one to reduce a question about the multiplicity of an ideal in a local ring to the same question when the ring is a domain.

This will be especially important when we relate the multiplicity of an ideal to the Rees valuation rings associated to the ideal.

Multiplicities

Proposition K5. Let $(R, \mathfrak{m}, k)$ be a local ring, $I \subseteq R$ an $\mathfrak{m}$-primary ideal, and $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules. Then $e(I, B) = e(I, A) + e(I, C)$.

Proof. For all $n \geq 1$, we have an exact sequence

$$0 \to A/(I^n B \cap A) \to B/I^n B \to C/I^n C \to 0,$$

and thus

$$\lambda(B/I^n B) = \lambda(A/(I^n B \cap A)) + \lambda(C/I^n C), \qquad (*)$$

for all n. Now, by the Artin-Rees lemma there exists $c > 0$ such that $I^n B \cap A \subseteq I^{n-c} A$, for all $n > c$. Thus,

$$\lambda(A/I^{n-c} A) \leq \lambda(A/I^n B \cap A)) \leq \lambda(A/I^n A), \qquad (**)$$

for all $n > c$. Applying $\lim_{n\to\infty} \frac{d!}{n^d} \cdot \lambda(-)$ to (**) shows that $\lim_{n\to\infty} \frac{d!}{n^d} \cdot \lambda(A/(I^n B \cap A))$ exists and equals $e(I, A)$.

Therefore, applying $\lim_{n\to\infty} \frac{d!}{n^d} \cdot \lambda(-)$ to equation (*) gives what we want. □

Corollary L5. Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d, $I \subseteq R$ an $\mathfrak{m}$-primary ideal and M a finitely generated R-module. Then, degree $P_{I,M}(x) = \dim(M)$. Thus, $e(I, M) = 0$ if and only if $\dim(M) < d$.

Proof. The second statement follows immediately from the first. For the first statement, we may mod out the annihilator of M and assume that the annihilator of M is zero.

We now have $\dim(M) = \dim(R)$ and an inclusion

$$R \hookrightarrow M \oplus \cdots \oplus M$$

.Set $C := M \oplus \cdots \oplus M$, so that C is a finite R-module. It follows from the proposition above that $e(I, R) \leq e(I, C)$.

Thus, the degree of $P_{I,C}(x)$ is greater than or equal to the degree of $P_I(x)$, which is $\dim(R)$.

On the other hand, we always have that degree $P_{I,C}(x) \leq \dim(R)$, so equality holds. Since $P_{I,C}(x)$ and $P_{I,M}(x)$ have the same degree, the proof is complete. □

Multiplicities

Remark. For the proof of the associativity formula, we need the following standard fact. If M is a finite module over the Noetherian ring R, then there exists a filtration

$$(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$$

such that each quotient $M_i/M_{i-1} \cong R/P_i$, for some $P_i \in \mathrm{Spec}(R)$.

To see this, let M' be maximal among all submodules of M admitting a filtration of the required type. The set of such modules in non-empty, since if $P \in \mathrm{Ass}(M)$, R/P is isomorphic to a submodule of M.

If $M' \neq M$, then we can extend the filtration one step beyond M' by considering a submodule of M/M' corresponding to R/P, for $P \in \mathrm{Ass}(M/M')$.