April 20: Integral closure of ideals, multiplicity, and the Rees multiplicity theorem

The Rees Multiplicity Theorem

In this section, we change direction entirely to focus on multiplicities in local rings, with the goal of proving the celebrated theorem of Rees, which states the following:

Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring and $J \subseteq I$ two $\mathfrak{m}$-primary ideals satisfying $e(J) = e(I)$. Then $\overline{J} = \overline{I}$.

Here we are writing $e(I)$ for the multiplicity if I.

Recall that if $(R, \mathfrak{m})$ is a local ring with $\dim(R) = d$, and $I \subseteq R$ is an $\mathfrak{m}$-primary ideal, one way to define $e(I)$ is as follows:

$$e(I) = \lim_{n \to \infty} \frac{d!}{n^d} \cdot \lambda(R/I^n),$$

where we use $\lambda(-)$ to denote the length of a finite length R-module.

We will develop the definition and basic properties of multiplicities in the next lecture.

Integral closure of ideals revisited

In order to prove the Rees theorem will will start with some preliminaries on integral closure and then present the standard background material on the multiplicity of an $\mathfrak{m}$-primary ideal in a local ring.

Definition and Comments. An integral domain V with quotient field K is a *valuation domain* if for every $x \in K$, either $x \in V$ or $x^{-1} \in V$. Note that a DVR W is easily seen to be a valuation domain, since every element of W has the form $u\pi^n$, for $u \in W$ a unit and $\pi \in W$ the uniformizing parameter of W.

The following hold for a valuation domain V:

(i) Every finitely generated ideal of V is principal. To see this, it suffices to show any two-generated ideal is principal, and to see this it suffices to see that if $a, b \in V$ are nonzero, then either $a \in bV$ or $b \in aV$. But, by definition, either $\frac{a}{b} \in V$ or $\frac{b}{a} \in V$, which gives what we want.

NOTE: A valuation domain does not have be a Noetherian. In fact, any Noetherian valuation domain is a DVR.

Integral closure of ideals revisited

(ii) A valuation domain has a unique maximal ideal. To see this, let $\mathfrak{m}_V$ denote the set of non-units of V. Clearly $va \in \mathfrak{m}_V$ for all $v \in V$ and $a \in \mathfrak{m}_V$. If $a, b \in \mathfrak{m}_V$ then $a \in bV$ or $b \in aV$, by the previous item.

Say, $a \in bV$, so $a = bv$, some $v \in V$. Then $a + b = (v + 1)b \in \mathfrak{m}_V$, so $\mathfrak{m}_V$ is closed under addition. Thus, $\mathfrak{m}_V$ is an ideal, and is therefore the unique maximal ideal of V.

(iii) V is integrally closed. To see this, suppose $x \in K$ is integral over V. We have an equation of the form:

$$x^n + v_1x^{n-1} + \cdots + v_n = 0,$$

with each $v_j \in V$. Since V is a valuation domain, either x or x^{-1} belongs to V. Suppose $x^{-1} \in V$. Multiply the equation above by x^{-n} to get

$$1 + v_1x^{-1} + \cdots + v_nx^{-n} = 0.$$

Solving for 1 in this equation, we can write $1 = vx^{-1}$, for some $v \in V$. This shows x^{-1} is a unit in V, and hence its inverse x is in V.

Integral closure of ideals revisited

(iv) Every ideal in a valuation domain is integrally closed. To see this, let $J \subseteq V$ be an ideal, and take $b \in \overline{J}$. Then b in integral over a finitely generated ideal $J_0 \subseteq J$.

By (i) J_0 is principal, and by (iii) V is integrally closed. Thus, $\overline{J_0} = J_0$, and hence $b \in J_0 \subseteq J$.

Our first goal is to characterize the integral closure of powers of an ideal in a Noetherian ring in terms of discrete valuation rings.

Proposition A5. Let R be a Noetherian domain with quotient field K and $J \subseteq R$ an ideal. Then

$$\overline{J} = \bigcap_V (JV \cap R) = (\bigcap_V JV) \cap R,$$

where the intersection runs through the DVRs between R and K.

Proof. Let $b \in \overline{J}$. From (iv) above $b \in JV$, for all V.

Conversely, suppose $b \notin \overline{J}$. We must find a DVR between R and K with $b \notin JV$.

Integral closure of ideals revisited

Suppose $J = (a_1, \ldots, a_d)R$, and set $S := R[\frac{a_1}{b}, \ldots, \frac{a_d}{b}]$. Set $L := (\frac{a_1}{b}, \ldots, \frac{a_d}{b})S$.

We claim $L \neq S$. Suppose $L = S$. Then there exists a polynomial $f(x_1, \ldots, x_d)$ with coefficients in R such that $f(\frac{a_1}{b}, \ldots, \frac{a_d}{b}) = 1$.

Note that if $x_1^{e_1} \cdots x_d^{e_d}$ is a monomial in $f(x_1, \ldots, x_d)$ of degree n, then $b^N \cdot (\frac{a_1}{b})^{e_1} \cdots (\frac{a_d}{b})^{e_d} \in b^{N-n}J^n$, for all $N \geq n$.

Thus, if N is the largest degree of a monomial in $f(x_1, \ldots, x_d)$ and we multiply $f(\frac{a_1}{b}, \ldots, \frac{a_d}{b}) = 1$ by b^N and bring b^N to the left hand side of the resulting equation, we have an equation of integral dependence of b on J, contrary to our choice of b. Thus, L is a proper ideal of S.

Now, take a prime ideal $P \subseteq S$ containing L. Then by Corollary G2, there exists a DVR V between S and its quotient field, which is K, such that $\mathfrak{m}_V \cap S = P$.

Thus, the elements $\frac{a_i}{b}$ are non-units in V. If b were in JV, then for a_i with $JV = a_iV$, we would have $b \in a_iV$. But then $\frac{b}{a_i} \in V$, a contradiction. Therefore $b \notin JV$, and the proof is complete. □

Remark. Let A be an integral domain, not necessarily Noetherian, with quotient field K. Take a prime ideal $P \subseteq A$. Via Zorn's lemma,there exists a valuation domain V (more than likely not a DVR) such that $\mathfrak{m}_V \cap A = P$.

The proof above, together with the comments above, show that $\overline{J} = \bigcap_V (JV \cap A)$, where the intersections runs through all valuation domains between A and K.

Corollary B5. Let R be a Noetherian domain and $I = (a_1, \ldots, a_d)R$ and ideal. Set $T_i = R[\frac{a_1}{a_i}, \ldots, \frac{a_d}{a_i}]$. Then, for all $n \geq 1$, $\overline{I^n} = \bigcap_{1 \leq i \leq d} (\overline{I^n T_i} \cap R)$.

Proof. Note: $IT_i = a_i T_i$, all i. Clearly $\overline{I^n} \subseteq \bigcap_{1 \leq i \leq d} (\overline{I^n T_i} \cap R)$.

Conversely, suppose $b \in \bigcap_{1 \leq i \leq d} (\overline{I^n T_i}. \cap R)$. Let V be a DVR between R and its quotient field.

If $IV = a_i V$, then $a_j \in a_i V$, for all $j \neq i$. Thus each fraction $\frac{a_j}{a_i} \in V$.

Therefore $T_i \subseteq V$. But now, $b \in \overline{I^n T_i} \subseteq \overline{I^n V} = I^n V$.

Since this holds for all DVRs between R and its quotient field, $b \in \overline{I^n}$, by Proposition A4. □

Theorem C5. Let R be a Noetherian domain with quotient field K and $I \subseteq R$ be an ideal. Then there exist finitely many DVRs $V_1, \dots, V_r$ between R and K such that for any $n \geq 1$, $\overline{I^n} = \bigcap_{i=1}^{r}(I^n V_i \cap R)$.

Proof. Let $I := (a_1, \dots, a_d)R$ and set $T_i := R[\frac{a_1}{a_i}, \cdots, , \frac{a_d}{a_i}]$, for all $1 \leq i \leq d$. Take $n \geq 1$, fix $1 \leq i \leq d$ and let T_i' denote the integral closure of T_i. Then T_i' is a Krull domain and thus, from our work in Section 2 we have:

(i) There exist finitely many height one primes $Q_1, \dots, Q_s \subseteq T_i'$ containing a_i^n, which are exactly the height one primes containing a_i.

(ii) $a_i^n T_i' = (a_i^n W_1 \cap T_i') \cap \cdots \cap (a_i W_s \cap T_i')$, where each $W_j = (T_i')_{Q_j}$.

(iii) Each W_j is a DVR.

Since $\overline{a_i^n T_i} = a_i^n T_i' \cap T_i$, we have

$$\overline{I^n T_i} = \overline{a_i^n T_i} = a_i^n T_i' \cap T_i = (a_i^n W_1 \cap \cdots \cap a_i^n W_s) \cap T_i.$$

Therefore,

$$\overline{I^n T_i} \cap R = (a_i^n W_1 \cap \cdots \cap a_i^n W_s) \cap R = (I^n W_1 \cap R) \cap \cdots \cap (I^n W_s \cap R).$$

If we do this for each i, and collect all of the resulting DVRs associated to each $a_i T_i'$, and call them $V_1, \dots, V_r$, then the theorem follows from Corollary B5.

Remark. The DVRs $V_1, \dots V_r$ constructed in the proof of Theorem C5 are called the Rees valuation rings of I, and are uniquely determined as the smallest collects of DVRs between R and K for which the conclusion of Theorem C5 holds.

We next want to improve the conclusion of Theorem C5 in the case that R is a local domain satisfying the dimension formula and the generators of I form a system of parameters. The next proposition is a special case of a result to E.D. Davis.

Proposition D5. Let $(R, \mathfrak{m}, k)$ be a local domain and $a_1 \dots, a_d$ a system of parameters. Fix $1 \leq i \leq d$ and set $T_i := R[\frac{a_1}{a_i}, \cdots , , \frac{a_d}{a_i}]$. Then $\mathfrak{m}T_i$ is a height one prime and the residue classes of $\frac{a_1}{a_i}, \dots, \widehat{i}, \dots, \frac{a_d}{a_i}$ are algebraically independent over k, i.e., $T_i/\mathfrak{m}T_i$ is isomorphic to a polynomial ring in $d-1$ variables over k.

Proof. It suffices to prove the case $i = 1$. Set $T := T_1$ and $S := R[x_2, \dots, x_d]$, the polynomial ring in $d-1$ variables over R. Let P denote the kernel of the natural ring homomorphism from S to T that takes each x_i to $\frac{a_i}{a_1}$, so $P \cap R = 0$.

Integral closure of ideals revisited

Let L denote the ideal of S generated by $a_1x_2 - a_2, \ldots, a_1x_x - a_d$, so that $L \subseteq P$.

We note: If we invert a_1, then S_{a_1} is the polynomial ring in $d-1$ variables over R_{a_1} and $T_{a_1} = R_{a_1}$. The induced ring homomorphism from $S_{a_1} \to T_{a_1}$ is now just obtained by evaluating any $g(x_2, \ldots, x_d) \in S_{a_1}$ at $\frac{a_2}{a_1}, \ldots, \frac{a_d}{a_1} \in R_{a_1}$.

The kernel of an evaluation map is alway just the expected kernel, in this case, $L_0 = (x_2 - \frac{a_2}{a_1}, \ldots, x_d - \frac{a_d}{a_1})S_{a_1}$.

Now, clearly $LS_{a_1} = L_0S_{a_1}$, while on the other hand, the kernel of the induced map is P_{a_1}.

Thus $P_{a_1} = L_{a_1}$, and hence $P = L_{a_1} \cap S$.

In other words, $f(x_2, \ldots, x_d) \in P$ if and only if $a_1^c \cdot f(x_2, \ldots, x_d) \in L$, for some $c \geq 1$.

Integral closure of ideals revisited

So: take $f(x_2, \dots, x_d) \in P$ and let $Q \subseteq S$ be a prime minimal over L. Then $a_1^c \cdot f(x_2, \dots, x_d) \in L \subseteq Q$. Suppose $a_1^c \in Q$. Then $a_1 \in Q$ and hence $a_1, \dots, a_d \in Q$. But this is a contradiction, since on the one hand $\mathrm{height}(Q) \leq d-1$, while on the other hand $a_1, \dots, a_d$ generate an ideal of height d in R, and hence also in S.

Thus, $a_1 \notin Q$, so $f(x_2, \dots, x_d) \in Q$. Therefore, $P \subseteq Q$, which shows that P is the unique minimal prime of L.

Now, since $\mathfrak{m}S$ contains L, we have $P \subseteq \mathfrak{m}S$ Thus $\mathfrak{m}T = \mathfrak{m}S/P$ is a prime ideal. In addition, for some $n \geq 1$,

$$\mathfrak{m}^n \subseteq (a_1, \dots, a_d)S = (a_1, L)S \subseteq (a_1, P)S,$$

which shows $\mathfrak{m}^n \subseteq a_1 T$.

Thus $\mathrm{height}(\mathfrak{m}T) = 1$, and in fact, $\mathfrak{m}T$ is the unique height one prime in T containing a_1.

Finally,

$$T/\mathfrak{m}T = (S/P)/(\mathfrak{m}S/P) \cong S/\mathfrak{m}S \cong k[x_2, \dots, x_d],$$

the polynomial ring in $d-1$ variables over k. □

The following proposition due to DK plays a key role in a theorem below concerning multiplicities.

Proposition E5. Let $(R, \mathfrak{m}, k)$ be a local domain and $I = (a_1, \ldots, a_d)R$ an ideal generated by a system of parameters. Assume R satisfies the dimension formula. Set $S := R[\frac{a_2}{a_1}, \cdots, , \frac{a_d}{a_1}]_{\mathfrak{m}R[\frac{a_2}{a_1}, \cdots, , \frac{a_d}{a_1}]}$ and let $Q_1, \ldots, Q_s$ be the height one primes in S'.

Then for all $n \geq 1$, $\overline{I^n} = (I^n V_1 \cap R) \cap \cdots \cap (I^n V_s \cap R)$, where $V_i := (S')_{Q_i}$, for each i.

Proof. By Theorem C5, we just have to show that $V_1, \ldots, V_s$ is the complete set of Rees valuation rings of I. For each $1 \leq i \leq d$, set $T_i := R[\frac{a_1}{a_i}, \cdots, , \frac{a_d}{a_i}]$, so that $S = (T_1)_{\mathfrak{m}T_1}$.

By Proposition D5, $\mathfrak{m}T_i$ is a height one prime. Let $U_i \subseteq T_i$ be the multiplicatively closed subset generated by $\frac{a_1}{a_i}, \ldots, \frac{a_d}{a_i}$.

Then

$$(T_i)_{U_i} = T_i[(\frac{a_1}{a_i})^{-1}, \ldots, (\frac{a_d}{a_i})^{-1}] = R[\frac{a_1}{a_i}, \ldots, \frac{a_d}{a_i}, \frac{a_i}{a_1}, \ldots, \frac{a_i}{a_d}].$$

Integral closure of ideals revisited

Let $2 \leq j \leq d$ and $i \neq 1$. If $j = i$, then $\frac{a_1}{a_j}, \frac{a_j}{a_1} \in (T_i)_{U_i}$. If $j \neq i$,

$\frac{a_j}{a_1} = \frac{a_j}{a_i} \cdot \frac{a_i}{a_1} \in (T_i)_{U_i}$ and $\frac{a_1}{a_j} = \frac{a_1}{a_i} \cdot \frac{a_i}{a_j} \in (T_i)_{U_i}$,

which shows that $(T_1)_{U_1} \subseteq (T_i)_{U_i}$.

The same argument shows $(T_i)_{U_i} \subseteq (T_1)_{U_1}$, and thus $(T_i)_{U_i} = (T_1)_{U_1}$, for all i.

By Proposition D5, $U_i \cap \mathfrak{m}T_i = \emptyset$, for all i, since the images of the elements $\frac{a_j}{a_i}$ in $T_i/\mathfrak{m}T_i$ are algebraically independent over k.

Thus $(T_i)_{\mathfrak{m}T_i} = ((T_i)_{U_i})_{\mathfrak{m}(T_i)_{U_i}}$ for all i, from which we infer $(T_i)_{\mathfrak{m}T_i} = S$, for all i.

Integral closure of ideals revisited

Now let V be a Rees valuation ring of I. Then for some $1 \leq i \leq d$, V is obtained by localizing T_i' at a height one prime Q containing a_i, so that $\mathfrak{m}_V = Q_Q$.

Thus, by Proposition D3, $Q \cap T_i \in \overline{A^*}(a_i T_i)$. Since R satisfies the dimension formula, T_i also satisfies the dimension formula, by Observation 1 following Corollary Q3. Thus, by Proposition L3, $\mathrm{height}(Q) = 1$.

Since $\mathfrak{m} T_i$ is the only height one prime in T_i containing a_i, $\mathfrak{m} T_i = Q \cap T_i$.

Thus $S := (T_1)_{\mathfrak{m} T_1} = (T_i)_{\mathfrak{m} T_i} \subseteq (T_i')_Q = V$. Since V is integrally closed $S' \subseteq V$.

Since $\mathfrak{m}_V \cap S = \mathfrak{m} S$, $\mathfrak{m}_V \cap S'$ must contract to $\mathfrak{m} S$, therefore $\mathfrak{m}_V \cap S' = Q_j$, for some $1 \leq j \leq s$.

It follows that $V_j \subseteq V$. However, there are no rings strictly between a DVR and its quotient field, so we must have $V_j = V$, which is what we want. □