April 6: Quasi-unmixedness and Ratliff'sTheorem, part 1

Preliminaries

The purpose of this section is to study quasi-unmixed local rings with the goal of proving a fundamental theorem due to Ratliiff, which gives equivalent conditions for a local ring $(R, \mathfrak{m})$ to be quasi-unmixed. Recall that R is said to be *quasi-unmixed* or *formally equi-dimensional* if $\dim(\widehat{R}/q) = \dim(\widehat{R})$, for all minimal primes $q \subseteq \widehat{R}$.

Here is Ratliff's Theorem, stated for integral domains.

Theorem A3. Let $(R, \mathfrak{m})$ be a local integral domain. The following statements are equivalent.

(i) R is quasi-unmixed.

(ii) R is universally catenary.

(iii) R satisfies the dimension formula.

To address the other conditions in Ratliff's theorem, we need a few definitions.

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Definition. Let S be a Noetherian ring.

(i) S is catenary if for all pairs of primes $P \subseteq Q \subseteq S$, all saturated chains of prime ideals between P and Q have the same length.

(ii) S is universally catenary if every finitely generated S-algebra is catenary.

(iii) If S is an integral domain, then S satisfies the dimension formula if for every finitely generated S-algebra T and prime ideal $Q \subseteq T$, we have:

$$\mathrm{height}(Q) + \mathrm{tr.deg}_{k(Q\cap S)} k(Q) = \mathrm{height}(Q \cap S) + \mathrm{tr.deg}_S T.$$

Several remarks are in order.

Remarks. (i) The conditions in Ratliff's theorem are not equivalent if R is an arbitrary local ring - for trivial reasons. For example, the ring $k[[x, y, z]]/(x) \cap (y, z)$ is a complete local ring and therefore is universally catenary, something we will see later in this section. On the other hand it is not equi-dimensional and since it is complete, it is not quasi-unmixed. If we assume that R is equi-dimensional, then conditions (i) and (ii) in Ratliff's theorem are equivalent. But the proof of this equivalence easily reduces to the domain case.

(ii) It turns out that the rings from algebraic geometry are all universally catenary. In the late 1940s and early 1950s, it was not known whether or not Noetherian rings in general were catenary or universally catenary. In the mid 1950s, Nagata gave an example of a Noetherian ring that was catenary, but not universally catenary.

(iii) If $S \subseteq T$ an extension of Noetherian domains and T is a finitely generated algebra over S, then the following *dimension inequality* always holds:

$$\mathrm{height}(Q) + \mathrm{tr.deg}_{k(Q\cap S)}k(Q) \leq \mathrm{height}(Q \cap S) + \mathrm{tr.deg}_S T.$$

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(iv) To invoke the dimension formula, one needs an extension of integral domains. One could make a definition in the case that S is not a domain, by requiring that S/q satisfies the dimension formula for all minimal primes $q \subseteq S$. Again, in order to have conditions (i) and (iii) in Ratliff's theorem equivalent, one would have to require that S be $\mathrm{equi\text{-}dimensional}$, and this case too reduces easily to the case that S is an integral domain.

(v) It is not difficult to see that if S is a Noetherian domain, then the dimension formula holds between S and $S[x]$, the polynomial ring over S. To see this, note that $\mathrm{tr.deg}_S S[x] = 1$. Take a prime $Q \subseteq S[x]$ and set $P = Q \cap S$. There are two cases to consider.

If $Q = PS[x]$, then $\mathrm{height}(P) = \mathrm{height}(Q)$ and $S[x]/Q = S/P[x]$, and thus $\mathrm{tr.deg}_{S/P} S[x]/Q = 1$. So the dimension formula holds between S and $S[x]$.

If $Q \neq PS[x]$, then $\mathrm{height}(Q) = 1 + \mathrm{height}(P)$ and $S[x]/Q$ is algebraic over S/P, so again, the required equality holds.

Primes associated to the integral closure of ideals

We now work towards a characterization of quasi-unmixed local rings obtained by studying *asymptotic sequences*, an integral closure analogue of regular sequences. Our first goal is to show that if $I \subseteq R$ is an ideal, then $\bigcup_{n\geq 1} \mathrm{Ass}(R/\overline{I^n})$ is finite. Throughout the remainder of this section, R denotes a Noetherian ring.

Lemma B3. Let S be a Noetherian ring and $J \subseteq S$ be an ideal. Then for $a \in S$, $a \in \overline{J}$ if and only if for all minimal primes $q \subseteq S$, the image of a in S/q belongs to $\overline{(J+q)/q}$.

Proof. The forward direction is clear. Suppose the image of a in S/q belongs to $\overline{(J+q)/q}$, for all minimal primes $q \subseteq S$. Then for each q there is an n (depending on q) and equation of the form

$$a^n + j_1 a^{n-1} + \cdots + j_n \equiv 0 \bmod q,$$

where each $j_i \in J^i$.

Taking the product of these equations yields an equation of the form

$$a^m + j_1 a^{m-1} + \cdots + j_m \equiv 0 \bmod N,$$

where each $j_i \in J^i$, and N denotes the nilradical of S.

Raising this last congruence to an appropriate power shows $a \in \overline{J}$. □

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Corollary C3. Let S be a Noetherian ring and $J \subseteq S$ be an ideal. If $P \in \mathrm{Ass}(S/\overline{J})$, then there is a minimal prime $q \subseteq P$ with $P/q \in \mathrm{Ass}\ S/\overline{J+q}$.

Proof. Without loss of generality, we assume S is local at P. Write $P = (\overline{J} : a)$, for some $a \notin \overline{J}$. Then, by Lemma B2, $a \notin \overline{(J+q)/q}$, for some minimal primes $q \subseteq S/q$. Thus, P/q consists of zerodivisors mod $\overline{(J+q)/q}$, which gives what we want. □

The next crucial proposition is a nice application of the Mori-Nagata theorem and properties of Krull domains.

Proposition D3. Let S be a Noetherian domain and $0 \neq a \in S$. Then $P \in \mathrm{Ass}\ (S/\overline{a^nS})$ for some $n \geq 1$ if and only if there exists a height one prime $Q \subseteq S'$ containing a such that $Q \cap S = P$.

In particular $\bigcup_{n\geq 1} \mathrm{Ass}\ (S/\overline{a^nS})$ is a finite set.

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Proof. The second statement follows immediately from the first. To prove the first statement, we may assume R is local at P.

Suppose $P \in \mathrm{Ass}\,(S/\overline{a^nS})$ for some $n \geq 1$ and write $P = (\overline{a^nS} : b)$, with $b \notin \overline{a^nS}$.

Let $Q_1, \ldots, Q_r$ be the height one primes in the Krull domain S' containing a, and write $a^nS' = C_1 \cap \cdots \cap C_r$, where each C_i is Q_i-primary.

Since $a^nS' \cap S = \overline{a^nS}$, $b \notin C_i$, some i.

But $Pb \subseteq a^nS' \subseteq C_i$, so we must have $P \subseteq Q_i$, since C_i is Q_i-primary.

Therefore $Q_i \cap S = P$.

The proof of the converse requires just minor tweaking of the proof of Lemma K2.

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Take a height one prime $Q \subseteq S'$ containing a. Since Q is minimal over aS', for all $q \in Q$, there exists $s \in S' \backslash Q$ such that $s \cdot q^h \in aS'$, for some h.

If we do this for the finitely many generators of P, it follows that there exists $s \in S' \backslash Q$, $t \geq 1$ and a ring $S \subseteq S_0 \subseteq S'$, such that $P^t \cdot s \subseteq aS_0$ and S_0 is a finite S-module.

Thus, for all $n \geq 1$, $\overline{P^{nt}} \cdot s^n \subseteq a^n S_0$. Let $0 \neq c \in S$ satisfy $c \cdot S_0 \subseteq S$. Then, $\overline{P^{nt}} \cdot (cs^n) \subseteq a^n S \subseteq \overline{a^n S}$, for all n.

If $cs^n \in \overline{a^n S}$ for all n, then $c \in a^n S'_Q$, for all n, since $s \notin Q$.

But then $c \in \bigcap_{n \geq 1} a^n S'_Q = 0$, since S'_Q is a DVR. This is a contradiction.

Thus $cs^n \notin \overline{a^n S}$, for some n, which implies $P \in \mathrm{Ass}\,(S/\overline{a^n S})$. □

Remark. Note that the last paragraph of the proof above shows that if S is a Noetherian domain and $0 \neq a \in S$, then $\bigcap_{n\geq 1} \overline{a^nS} = 0$. This is extended to arbitrary ideals below.

Corollary E3. Let S be a Noetherian ring and $a \in S$ be a non-zerodivisor. Then $\bigcup_{n\geq 1} \mathrm{Ass}\ (S/\overline{a^nS})$ is finite.

Proof. Immediate from C3 and D3.

We need one more lemma before we can show that $\bigcup_{n\geq 1} \mathrm{Ass}(R/\overline{I^n})$ is finite.

Lemma F3. Let $I \subseteq R$ be an ideal and $\mathcal{R} := R[It, t^{-1}]$ denote the extended Rees aring of R with respect to I. Then for all $n \geq 1$:

(i) $\overline{I^n} = \overline{t^{-n}\mathcal{R}} \cap R$.

(ii) The vth graded component of $\overline{t^{-n}\mathcal{R}}$ is $(I^v \cap \overline{I^{n+v}})t^v$, for all v.

Proof. For (i), take $a \in R$. Suppose $a \in \overline{I^n}$. Then there exists an equation of the form

$$a^s + i_na^{s-1} + i_{2n}a^{s-2} + \cdots + i_{sn} = 0,$$

where each $i_{jn} \in I^n$. Multiply this equation by t^{sn} to get

$$(\frac{a}{t^{-n}})^s + i_nt^n(\frac{a}{t^{-n}})^{s-1} + i_{2n}t^{2n}(\frac{a}{t^{-n}})^{s-2} + \cdots + i_{sn}t^{sn} = 0. \quad (*)$$

This shows $\frac{a}{t^{-n}}$ is integral over $\mathcal{R}$, so $a \in \overline{t^{-n}\mathcal{R}}$.

Primes associated to the integral closure of ideals

Conversely, if $a \in \overline{t^{-n}\mathcal{R}} \cap R$, then $\frac{a}{t^{-n}}$ is integral over $\mathcal{R}$. By comparing terms of the same degree in an equation of integral dependence of $\frac{a}{t^{-n}}$ over $\mathcal{R}$, we may work backwards from an equation like (*) to show $a \in \overline{I^n}$. □

The proof of (ii) is almost the same. Suppose $ct^v \in \overline{t^{-n}\mathcal{R}}$. Then clearly $c \in I^v$.

On the other hand, $\frac{ct^v}{t^{-n}} = ct^{v+n}$ is integral over $\mathcal{R}$. Thus, there exists an equation of the form

$$(ct^{v+n})^s + f_1(ct^{v+n})^{s-1} + \cdots + f_s = 0,$$

with $f_i \in \mathcal{R}$. Taking the coefficient of $t^{s(n+v)}$ in this equation gives

$$c^s + j_1 c^{s-1} + \cdots + j_s = 0,$$

where each $j_i \in I^{i(v+n)}$. Thus, $c \in \overline{I^{n+v}}$. □

The converse of (ii) is similar.

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Theorem G3. Let R be a Noetherian ring, $I \subseteq R$ an ideal and $\mathcal{R}$ the extended Rees ring of R with respect to I. For a prime $P \subseteq R$, $P \in \overline{A^*}(I)$ if and only if there exists $Q \in \overline{A^*}(t^{-1}\mathcal{R})$ with $Q \cap R = P$. In particular, for any ideal $I \subseteq R$, $\bigcup_{n\geq 1} \mathrm{Ass}(R/\overline{I^n})$ is finite.

Proof. In light of Corollary E3, it suffices to prove the first statement. Without loss of generality we may assume R is local at P. Suppose $P = (\overline{I^n} : c)$, for some $n \geq 1$ and $c \notin \overline{I^n}$. By the previous lemma, $c \notin \overline{t^{-n}\mathcal{R}}$.

Thus P consists of zero divisors on $\mathcal{R}/\overline{t^{-n}\mathcal{R}}$. It follows that $P\mathcal{R} \subseteq Q$, for some $Q \in \mathrm{Ass}\ (\mathcal{R}/\overline{t^{-n}\mathcal{R}})$. Thus, $Q \cap R = P$.

Conversely, suppose $Q \in \overline{A^*}(t^{-1}\mathcal{R})$. Write $Q = (\overline{t^{-n}\mathcal{R}} : ct^v)$.

Then Pct^v belongs to the degree v component of $\overline{t^{-n}\mathcal{R}}$, which, by the previous lemma is $(\overline{I^{n+v}} \cap I^y)t^v$.

Thus, $Pc \in \overline{I^{n+v}}$. Since $ct^v \notin \overline{t^{-n}\mathcal{R}}$, $c \notin \overline{I^{n+v}}$. This implies $n + v \geq 1$, thus, $P \in \overline{A^*}(I)$, as required. □

Primes associated to the integral closure of ideals

Remarks. (i) We denote the finite set of prime ideals in Theorem G3 by $\overline{A^*}(I)$. Note that $x \in R$ is a zerodivisor modulo $\overline{I^n}$ for some n if and only if $x \in P$, for some $P \in \overline{A^*}(I)$.

(ii) The proof of the Theorem G3 can be adapted to show the following: If $R \subseteq S$ are Noetherian rings, and $J \subseteq S$ is an ideal, then, if $P \in \mathrm{Ass}\ R/(J \cap R)$, there exists $Q \in \mathrm{Ass}\ S/J$ such that $Q \cap R = P$.

(iii) There is a stronger version of Theorem G3. Ratliff has shown that if $\mathrm{height}(I) > 0$, then the sets $\mathrm{Ass}\ R/\overline{I} \subseteq \mathrm{Ass} R/\overline{I^2} \subseteq \cdots$ form an ascending chain. Since the union of these set is finite, this increasing chain of sets must stabilize and we have that there exists an n_0 such that $\bigcup_{n\geq 1} \mathrm{Ass}\ R/\overline{I^n} = \mathrm{Ass}\ R/\overline{I^{n_0}}$.

(iv) The stronger statement in (iii) is an integral closure analogue of a theorem due to M. Brodmann who showed that for all finitely generated R-modules M and ideals $I \subseteq R$, $\mathrm{Ass}\ (M/I^nM)$ is stable for n sufficiently large.

However, the sets $\mathrm{Ass}\ (M/I^nM)$ need not be an increasing set of prime ideals.

When $M = R$, we will denote this stable value $A^*(I)$. A theorem of Ratliff shows that $\overline{A^*}(I) \subseteq A^*(I)$.

Asymptotic sequences

Definition. A sequence of elements $x_1, \ldots, x_r \in R$ is said to be an asymptotic sequence if for each $1 \leq i \leq r$, x_i does not belong to any prime ideal in $\overline{A^*}((x_1, \ldots, x_{i-1})R)$. In other words, for all i, x_i is not a zerodivisor modulo $\overline{(x_1, \ldots, x_{i-1})^n R}$, for all n.

Remarks. (i) Asymptotic sequences in the form above were defined independently Ratliff and D. Katz. Earlier, Rees had defined the notion of an asymptotic sequence over I, for an ideal I contained in a local ring. His definition was that x_i is not a zero divisor modulo $\overline{(I, x_i, \ldots, x_{i-1})^n R}$, for all n and all i. Rees used this concept to improve an earlier inequality of Burch that related analytic spread of an ideal $I \subseteq R$ to a difference between the dimension of R and the the depths of the modules R/I^n.

(ii) Ratliff and DK studied properties of asymptotic sequences, discarding the ideal I. They independently proved that a local ring is quasi-unmixed if and only if some (every) system of parameters forms an asymptotic sequence. This theorem will be our next main goal. Using this result one can give a natural proof of Ratliff's theorem, once one knows a little about how the dimension formula is related to the universally catenary property.

Asymptotic sequences

(iii) A regular sequence is an asymptotic sequence, though this is not obvious from the definitions. However, this is clear in the case of a single element, because x is the first element in a regular sequence if and only if x is a non-zerodivisor, while x is is the first element in an asymptotic sequence if and only if $\mathrm{height}(xR) = 1$, since $\overline{A^*}(0)$ is the set of minimal prime ideals of R.