April 13: Quasi-unmixedness and Ratliff'sTheorem, part 4 and Fibers of Ring Homomorphisms

Quasi-unmixed local rings

We are closing in on the last step in Ratliff's Theorem, namely that a quasi-unmixed local domain is universally catenary. The proof of the following Proposition is greatly facilitated by the use of asymptotic sequences.

Proposition U3. Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring. Let $P \subseteq R$ be a prime ideal.

(i) $\dim(R/P) + \text{height}(P) = \dim(R)$.

(ii) R/P is quasi-unmixed.

(iii) R_P is quasi-unmixed.

(iv) R is catenary.

Proof. Let $x_1, \ldots, x_r$ be an asymptotic sequence of maximal length from P. Then, there exists $P_0 \supseteq P$ with $P_0 \in \overline{A^*}((x_1, \ldots, x_r)R)$. By Theorem M3, there exists $Q \subseteq \widehat{R}$ with $Q \in \overline{A^*}((x_1, \ldots, x_r)\widehat{R})$ with $Q \cap R = P_0$.

Moreover, there exists $z \subseteq \widehat{R}$, a minimal prime, so that $Q_z \in \overline{A^*}((x_1, \ldots, x_r)\widehat{R}_z)$. On the one hand, by Proposition O3, $\text{height}(Q_z) \leq r$, since R_z satisfies the dimension formula.

On the other hand, by Theorem H3 (iii), $\text{height}(Q_z) \geq r$. Thus, $\text{height}(Q)_z = r$.

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Since $\widehat{R}$ is catenary,

$$r = \mathrm{height}(Q_z) = \dim(\widehat{R}/z) - \dim(\widehat{R}/Q) = \mathsf{dim}(\widehat{R}) - \dim(\widehat{R}/Q),$$

since R is quasi-unmixed. Therefore,

$$\dim(R/P) = \dim(\widehat{R}/P\widehat{R}) \geq \dim(\widehat{R}/Q) = \dim(\widehat{R}) - r \geq \dim(R) - \mathrm{height}(P).$$

Thus $\dim(R/P) + \mathrm{height}(P) \geq \dim(R)$. Since $\dim(R/P) + \mathrm{height}(P) \leq \dim(R)$ always holds, (i) follows.

Moreover, this shows $r = \mathrm{height}(P)$ and $\dim(\widehat{R}/Q) = \dim(R/P)$. In addition, since $\mathrm{height}(P_0) \leq r$, by Proposition O3, $P_0 = P$.

For (ii), let P and $x_1, \ldots, x_r \in P$ be as in (i). Then P is minimal over $(x_1, \ldots, x_r)R$.

Now assume Q is minimal over $P\widehat{R}$. Then Q is minimal over $(x_1, \ldots, x_r)\widehat{R}$ and thus belongs to $\overline{A^*}((x_1, \ldots, x_r)\widehat{R})$.

By what we have shown in (i), it follows that $\mathsf{dim}(\widehat{R}/Q) = \dim(R) - r = \dim(R/P)$, so R/P is quasi-unmixed.

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Now, since $r = \mathrm{height}(P)$, upon localizing R at P, $x_1, \ldots, x_r$ becomes an asymptotic sequence of length $\dim(R_P)$, so R_P is quasi-unmixed by Theorem P3. This gives (iii).

Finally, suppose R is $\mathrm{quasi\text{-}unmixed}$. Take $P \subseteq Q$ prime ideals.

We have to check the height condition in Observation 2 above. We may localize R at Q.

But then R_Q is quasi-unmixed, by (iii), and by part (i), the required height condition holds. Thus, R_Q, and hence R, is catenary.

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Our last step requires us to show that if R is a quasi-umixed local ring and T is a polynomial ring in finitely many variables over R, then T is locally quasi-unmixed.

Lemma V3. Let S be a Noetherian ring, $J \subseteq S$ an ideal and $R[x]$ the polynomial ring in one variable over R. Then $\overline{J[x]} = \overline{J}[x]$.

Proof. Write $\mathcal{R}$ for the extended Rees algebra of R with respect to J and note that $\mathcal{R}[x]$ is the Rees algebra of $R[x]$ with respect to $J[x]$.

Suppose $u(x) = u_n x^n + u_{n-1}x^{n-1} + \cdots + u_0$ belongs to $\overline{J[x]}$. By Lemma F3, $u(x) \in \overline{t^{-1}\mathcal{R}[x]}$.

Thus, $\frac{u(x)}{t^{-1}}$ is integral over $\mathcal{R}[x]$. Therefore, each $\frac{u_i}{t^{-1}}$ is integral over $\mathcal{R}$ for each i, and thus $u_i \in \overline{t^{-1}\mathcal{R}}$, for each i.

By Lemma F3, $u_i \in \overline{J}$ for all i, and hence, $u(x) \in \overline{J}[x]$, as required. □

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Corollary W3. Let $R[x]$ be the polynomial ring in one variable over R. If $x_1, \ldots, x_r$ is an asymptotic sequence in R, then $x_1, \ldots, x_r$ is an asymptotic sequence in $R[x]$.

Proof. For any ideal $I \subseteq R$, $Q \in \mathrm{Ass}(R[x]/I[x])$ if and only if there exists $P \in \mathrm{Ass}(R/I)$, with $Q = P[x]$.

Thus, in light of the previous lemma, for any ideal $J \subseteq R$, $Q \in \overline{A^*}(J[x])$ if and only if $Q = P$, for some $P \in \overline{A^*}(J)$.

The corollary now follows from the definition of asymptotic sequence. $\square$

Theorem X3. Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring and $T := R[x_1, \ldots, x_n]$ be the polynomial ring in n variables over R. Then for any prime ideal $Q \subseteq T$, T_Q is quasi-unmixed.

Proof. We induct on n. Suppose $n = 1$ and $Q \subseteq T$. Write $P := Q \cap R$. Then T_P is the polynomial ring in one variable over R_P and $(T_P)_Q = T_Q$. By Proposition U3, R_P is quasi-unmixed. Thus, we may begin again assuming $Q \cap R = \mathfrak{m}$.

Since R is quasi-unmixed, by Theorem P3, every system of parameters forms an asymptotic sequence. Let $x_1, \ldots, x_d \in R$ be a system of parameters, where $d := \dim(R)$.

Then: (i) By Corollary W3, $x_1, \ldots, x_d$ remain an asymptotic sequence in $R[x]$

(ii) $\overline{A^*}((x_1, \ldots, x_d)R) = \mathfrak{m}$.

Moreover, the proof of Corollary W3 shows $\overline{A^*}((x_1, \ldots, x_d)[x]) = \mathfrak{m}[x]$.

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Now, since $Q \cap R = \mathfrak{m}$, we have two cases.

$Q = \mathfrak{m}[x]$: Then $\dim(T_Q) = d$, and $x_1, \ldots, x_d \in Q$ is a system of parameters in T_Q forming an asymptotic sequence. Thus, by Theorem P3, T_Q is quasi-unmixed.

$Q \neq \mathfrak{m}[x]$: Then $Q = (\mathfrak{m}, f(x))T$, where $f(x)$ is a monic polynomial which is irreducible over $R/\mathfrak{m}$. Thus $f(x) \notin \mathfrak{m}[x] = \overline{A^*}((x_1, \ldots, x_d)[x])$.

Therefore, $x_1, \ldots, x_d, f(x)$ form an asymptotic sequence in T, and also in T_Q. Since $\dim(T_Q) = d + 1$, these elements are also a system of parameters in T_Q.

Thus, T_Q is quasi-unmixed, by Theorem P3.

Now suppose $n > 1$. Let $Q \subseteq T$ be a prime ideal and set $P := Q \cap S$, where $S = R[x_1, \ldots, x_{n-1}]$. By induction, S_P is quasi-unmixed.

Since T_P is the polynomial ring in one variable over S_P, the $n = 1$ case gives that $(T_P)_Q = T_Q$ is quasi-unmixed, and the proof is complete. $\square$

Here is the last component of Ratliff's Theorem.

Corollary Y3. Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring. Then R is universally catenary.

Proof. It is enough to show that if T is a polynomial ring in finitely many variables over R, then T is catenary. For this, it suffices to show that T_Q is catenary for every prime $Q \subseteq T$.

By Theorem $X3$, T_Q is quasi-unmixed and by Proposition U3, T_Q is catenary. □

For the sake of completeness, we put things all together.

Theorem (Ratliff). Let $(R, \mathfrak{m})$ be a local integral domain. The following statements are equivalent.

(i) R is quasi-unmixed.

(ii) R is universally catenary.

(iii) R satisfies the dimension formula.

Proof. (i) implies (ii) by Corollary Y3. (ii) implies (iii) by Proposition R3. (iii) implies (i) by Corollary Q3. □

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Here are two applications of the main results of this chapter.

Corollary Z3. Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring. Then:

(i) For any finitely generated R-algebra S and primes $q \subseteq Q \subseteq S$, $(S/q)_Q$ is quasi-unmixed.

(ii) $R[[x]]$, the formal power series ring over R, is quasi-unmixed.

Proof. There are a couple of ways to see (i), given everything we have done. For example, by Ratliff's theorem, R is universally catenary, so S is universally catenary, and hence $(S/q)_Q$ is universally catenary, and therefore quasi-unmixed.

Alternately, one can use the fact that if R is quasi-unmixed, then so is R/P for any prime $P \subseteq R$.

For (ii) one follows the ideas in the proof of Theorem X3 to show that if $a_1, \dots, a_d$ is a system of parameters in R forming an asymptotic sequence, then $a_1, \dots, a_d, x$ is a system of parameters forming an asymptotic sequence in $R[[x]]$. □

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We close this section with an amusing observation about height one primes in the integral closure of a local domain.

Observation. Let $(R, \mathfrak{m})$ be a local domain. Then there are only finitely many height one primes $Q \subseteq R'$ such that $\mathrm{height}(Q \cap R) > 1$.

Proof. Suppose $\{Q_n\}_n$ is an infinite set of height one primes in R' with $\mathrm{height}(Q_n \cap R) > 1$, for all n. For each Q_i, we can take a non-zero element $a_i \in Q_i$. By Proposition D3, $Q_i \in \overline{A^*}((a_i R))$.

By Theorem H3, Q_i lifts to a prime $P_i \in \overline{A^*}((a_i \widehat{R}))$. Each of these in turn have the property that $(P_i)_z \in \overline{A^*}((a_i \widehat{R}_z))$, for some minimal primes $z \in \widehat{R}$.

Proposition L3 gives $\mathrm{height}(P_i/z) = 1$.

We are now in the following situation: we have infinitely many primes $P_i \subseteq \widehat{R}$ with height greater than one (since each Q_i has height greater than one), and each P_i has height one modulo some minimal prime.

This cannot happen.

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Otherwise: since there are finitely many minimal primes in $\widehat{R}$, there must exist two minimal primes z_1, z_2 and infinitely many P_j such that $\mathrm{height}(P_j/z_1) > 1$ and $\mathrm{height}(P_j/z_2) = 1$, for all j.

Take $b \in z_1 \backslash z_2$. Then the non-zero principal ideal $b\widehat{R}_{z_2}$ is contained in infinitely many height one primes, which is a contradiction, since an ideal in a Noetherian ring has only finitely many minimal primes. □

Fibers of Ring Homomorphisms

In this section we present a few definitions and properties of the fiber of a ring map as they relate to the definition of an excellent local ring. Unless noted otherwise, all rings are Noetherian.

Definitions. Let $\phi : R \to S$ be a ring homomorphism. Tensor products are taken over R.

(i) For $p \subseteq R$ a prime ideal, the *fiber of ϕ over p* is the $k(p)$-algebra $k(p) \otimes S$. Note that since $k(p)$ is just the ring R_U/p_U where $U = R \backslash p$, the fiber over p is just S_U/pS_U.

(ii) When R is local and $S = \widehat{R}$, the fibers of ϕ are called the *formal fibers* of R.

(iii) If R is local with residue field k, $k \otimes S$ is the *closed fiber* of the map ϕ.

(iv) If R is an integral domain, and $p = (0)$, so that $k(p) = K$, the quotient field of R, then $K \otimes S$ is called the *generic fiber* of ϕ.

Remark. The description of $k(p) \otimes S$ in (i) shows that the prime ideals in the fiber of ϕ over p correspond to the prime ideals in S contracting to p. In fact, as topological spaces, one can show that $\mathrm{Spec}(k(p) \otimes S)$ is homeomorphic to the set of primes $P \in \mathrm{Spec}(S)$ with $P \cap R = p$.

Example 1. Suppose R is a Noetherian ring and $S = R[x_1, \ldots, x_n]$ is the polynomial ring in n variables over R. Take $p \subseteq R$ any prime ideal.

Then the fiber of the inclusion map $R \subseteq S$ is just $k(p)[x_1, \ldots, x_n]$. Thus, the fibers of this map look essentially the same, except the coefficient fields $k(p)$ can differ.

However, the dimension of each fiber is the same, namely n. S is certainly faithfully flat over R, but this alone is not enough to insure that the fibers of the inclusion map all have the same dimension.

Example 2. Let k be a field, and $R = \mathbb{Q}[y, z]$ be the polynomial ring in two variables over $\mathbb{Q}$. Then the formal power series ring $R[[x]]$ is faithfully flat over R. We now show that the fibers of the inclusion map $R \subseteq R[[x]]$ can have different dimensions.

We need to use the following fact: There exist two power series $f(x), g(x) \in \mathbb{Q}[[x]]$ that are algebraically independent over $\mathbb{Q}$.

In fact, the quotient field $\mathbb{Q}((x))$ of $\mathbb{Q}[[x]]$ has infinite transcendence degree over $\mathbb{Q}$.

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To see this, suppose the transcendence degree of $\mathbb{Q}((x))$ over $\mathbb{Q}$ were finite. Then we could find $f_1, \ldots, f_d \in \mathbb{Q}((x))$ such that $\mathbb{Q}((x))$ is algebraic over $K := \mathbb{Q}(f_1, \ldots, f_d)$. But K is a countable field, and since an algebraic extension of a countable field is countable, that would imply that $\mathbb{Q}((x))$ is countable.

$\mathbb{Q}[[x]]$ is clearly uncountable. Thus, $\mathbb{Q}((x))$ has infinite (uncountable!) transcendence degree over $\mathbb{Q}$, so we may choose $f(x), g(x)$ as above.

Now define a ring homomorphism $\alpha : R[[x]] \to \mathbb{Q}[[x]]$ by sending sending $\mathbb{Q}$ to itself, y to $f(x)$, z to $g(x)$ and x to itself. Note that this ring map exploits the fact that R is isomorphic to $\mathbb{Q}[f(x), g(x)]$.

Let P be the kernel of α. Then $y - f(x)$ and $z - g(x) \in P$. This forces P to have height 2.

But $P \cap R = (0)$, which shows that the generic fiber of the inclusion $R \subseteq R[[x]]$ has dimension two.

On the other hand, if we set $\mathfrak{m} = (y, z)R$, then $R[[x]]/\mathfrak{m}R[[x]] = \mathbb{Q}[[x]]$ is one dimensional, so the fiber over $\mathfrak{m}$ has dimension one. □