April 15: Fibers of Ring Homomorphisms

Fibers of Ring Homomorphisms

We begin with the following proposition, which gives some information about fibers of a ring homomorphism.

Proposition A4. Let $\phi : R \to S$ be a ring homomorphism. For $P \subseteq S$, set $p := P \cap S$. Then

$$\mathrm{height}(P) \leq \mathrm{height}(p) + \dim(k(p) \otimes S).$$

Equality holds if the going down property holds between R and S.

Proof. By localizing at P, R be comes local at p, and neither the heights or dimensions in question change. So we may assume that R is local at p and S is local at P.

Set $d := \dim(R)$ and $t := \dim(S/pS)$. Let $x_1, \ldots, x_d \in R$ be a system of parameters and $y_1, \ldots, y_t \in S$ be such that their images in S/pS form a system of parameters. Then $P^c \subseteq \underline{y}S + pS$ and $p^d \subseteq \underline{x}R$, for some $c, d \geq 1$. Then $P^{c+d} \subseteq (\underline{x} + \underline{y})S$. This shows $\dim(S) \leq d + t$, which gives the first statement.

Now suppose the going down property holds between R and S. Let $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_t = P$ be a saturated chain of primes containing pS. Note $P_0 \cap R = p$.

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Now let $p_0 \subsetneq \cdots \subsetneq p_d = p$ be a saturated chain in R. Then, by the going down property, in S, there is a chain of primes $P_0 = Q_d \supsetneq \cdots \supsetneq Q_0$ with $Q_i = p_i$, for all i.

This gives a chain of primes of length $d + t$ in S. Thus, $\dim(S) = d + t$, which is what we want. □

Proposition B4. Let $\phi : R \to S$ be a ring homomorphism so that S is faithfully flat over R. Then:

(i) ϕ is injective.

(ii) The going down and lying over properties hold between R and S. In particular, equality holds in Proposition A4.

(iii) $\mathrm{height}(I) = \mathrm{height}(IS)$, for all ideals $I \subseteq R$.

Proof. For (i) suppose $a \in R$ is non-zero. We have an exact sequence $0 \to aR \to R$. If we tensor with S (via ϕ), the sequence $0 \to (aR) \otimes S \to R \otimes S = S$ stays exact, by the flatness of S over R. The image of $a \otimes 1_S$ under this map is just $\phi(a)$. If $\phi(a) = 0$, then $(aR) \otimes S = 0$, which contradicts the faithfully flat property.

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For (ii), let $p \subseteq R$ be a prime ideal. Then the fiber $k(p) \otimes S$ is non-zero. Thus, $\mathrm{Spec}(k(p) \otimes S)$ is non-empty, so by our comments above, there exists a prime $P \subseteq S$ with $P \cap R = p$. In other words, the lying over property holds. Now suppose P_2 is a prime ideal in S, and set $p_2 := P_2 \cap R$ and suppose we have a prime $p_1 \subsetneq p_2$. Localizing at P_2 preserves flatness (by transitivity of flatness), so we may assume S is local at P_2. Since p_2 is the only maximal ideal of R and $p_2 S \neq S$, the extension is also faithfully flat. By lying over, there is a prime $P_1 \subseteq S$ with $P_1 \cap R = p_1$. Since S is local at P_2, $P_1 \subseteq P_2$, so the going down property holds.

For part (iii), let $p \subseteq R$ be a prime ideal and take $P \subseteq S$ a prime minimal over pS. Again, we may localize at P and assume that S is local at P and faithfully flat over R. By part (ii), the going down property holds, so by Proposition A,

$$\mathrm{height}(P) = \mathrm{height}(p) + \dim(k(p) \otimes S) = \mathrm{height}(p) + \dim(S/pS) = \mathrm{height}(p) + 0 = \mathrm{hei}$$

This argument shows $\mathrm{height}(I) = \mathrm{height}(IS)$. Indeed, if p is minimal over I having the same height as I, then the above shows $\mathrm{height}(IS) \leq \mathrm{height}(I)$. On the other hand, starting with P minimal over IS, P is minimal over pS, for $p = P \cap S$, so the argument shows $\mathrm{height}(IS) \leq \mathrm{height}(I)$, and therefore, $\mathrm{height}(I) = \mathrm{height}(IS)$. □

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Here is a proposition that sheds some light on the dimension of fibers. Note that, in general, the going up property does not holds between R and a polynomial ring or a power series over R.

Part (ii) of the example above shows that the going up property fails for power series rings, and that part (i) of the proposition below can fail in a faithfully flat extension, while if R is a DVR with uniformizing parameter π, going up fails for the extension $R \subseteq R[x]$, even though the fibers all have the same dimension.

To see this, note that $(\pi x - 1)R[x]$ is a maximal ideal in the polynomial ring contracting back to zero. If we take the chain $(0) \subseteq (\pi)$ we cannot lift it to a chain in $R[x]$ starting with $(px - 1)$, since the latter is a maximal ideal.

Proposition C4. Let $\phi : R \to S$ be a ring homomorphism between Noetherian rings. Let $q \subseteq p \subseteq R$ be prime ideals.

(i) If the going up holds, then $\dim(k(q) \otimes S) \leq \dim(k(p) \otimes S)$.

(ii) If going down holds, $\dim(k(p) \otimes S) \leq \dim(k(q) \otimes S)$.

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Proof. For (i), let $r := \dim(k(q) \otimes S)$. Then there exists a chain of distinct primes $Q_0 \subseteq \cdots \subseteq Q_r$ in S such that for each i, $Q_i \cap R = q$. Suppose $s := \operatorname{height}(p/q)$. Then in R, there exists a chain of distinct primes $q = p_0 \subseteq \cdots \subseteq p_s = p$. Since $p_0 = q$ by the going up property, we can lift this chain in R to a chain to $Q_r \subseteq \cdots \subseteq Q_{r+s}$, where each $Q_{r+j} \cap R = p_j$. Applying Proposition B4 to the induced homomorphism $R/q \to S/qS$, we have:

$$r + s \leq \operatorname{height}(Q_{r+s}/qS) \leq \operatorname{height}(p/q) + \dim(k(p/q) \otimes S/qS),$$

so $r \leq \dim(k(p/q) \otimes S/qS) = \dim(k(p) \otimes S)$, which is what we want.

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For (ii), we first note that if the conclusion of part (ii) holds when $\mathrm{height}(p/q) = 1$ then it holds in general. For if $q \subseteq p \subseteq p'$ with $\mathrm{height}(p/q) = 1 = \mathrm{height}(p'/p)$, then the longest chain of primes in S lying over p' is less than or equal to the longest chain of primes in S lying over p, which is less than or equal to the longest chain of primes in S lying over q, by the height one case. Iterating this shows we may assume $\mathrm{height}(p/q) = 1$.

Set $r := \dim(k(p) \otimes S)$. If $r = 0$, there is nothing to prove. Now, suppose $r > 0$ and let $P_0 \subseteq \cdots \subseteq P_r$ be a chain of distinct primes in S with $P_j \cap R = p$, for all j.

We need to find a chain of distinct primes $Q_0 \subseteq \cdots \subseteq Q_r$ in S, so that $Q_j \cap R = q$, for all j. For this we will use the following fact: If T is a Noetherian domain and C is a prime ideal in T having height greater than one, then C contains infinitely many height one primes of T.

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Now, by going down, there exist $Q_0 \subsetneq P_0$ such that $Q_0 \cap R = q$. Take $x \in p \backslash q$. We apply the fact above to $T := S/Q_0$ and its prime P_1/Q_0, which has height greater than one.

The fact above implies that there exists a height one prime contained in P_1/Q_0 not containing the image of x, since the image of x in T is contained in only finitely many height one primes This prime corresponds to a prime Q_1 in S containing Q_0, properly contained in P_1.

Since x is not in Q_1, we can't have $Q_1 \cap R = p$ and since there are no primes in R between q and p, we must have $Q_1 \cap R = q$. We can now apply the same process in S/Q_1 to the prime P_2/Q_1 which has height greater one. There is a height one prime in S/Q_1 contained in P_2/Q_1 not containing the image of x. As before, this corresponds to a prime Q_2 properly containing Q_1, which satisfies $Q_2 \cap R = q$. Continuing in this fashion, we can create a chain of primes of length r in S where each element of the chain contracts to q. This complete the proof of the proposition. □

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Remark Let R be a Noetherian ring.
(i) R satisfies Serre's condition S_n if for all $P \in \mathrm{Spec}(R)$, $\mathrm{depth}(R_P) \geq \min\{n, \dim(R_P)\}$. Thus, for example, a ring is Cohen-Macaulay if and only if it satisfies S_n for all $n \leq \dim(R)$.

(ii) R satisfies Serre's condition R_n if for all $P \in \mathrm{Spec}(R)$, with $\mathrm{height}(P) \leq n$, R_P is a regular local ring. A ring is regular if and only if it satisfies R_n for all n.

Comments. (i) R is reduced if and only if R satisfies R_0 and S_1. The conditions clearly hold if R is reduced. Suppose the conditions R_0 and S_1 fold. The condition S_1 implies that the associated primes of zero have height zero, i.e., are the minimal primes of R.

The R_0 condition implies that R_q is a field for each minimal prime $q \subseteq R$, and hence $q_q = 0$, for all minimal primes q. Together these conditions give $(0) = q_1 \cap \cdots \cap q_s$, where the q_i are the minimal primes of R. Therefore, R is reduced.

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(ii) Even though we have been considering integrally closed domains, the ring R does not have to be an integral domain to be integrally closed. We say that R is integrally closed (as a ring) if R equals the integral closure of R in its total quotient ring.

Note however, that if R is integrally closed, then either R is its total quotient ring or R must be reduced - since if $a \in R$ satisfies $a^c = 0$, then for any non-zerodivisor s in R, $\frac{a}{s}$ is an element in the total quotient ring of R, integral over R, yet not in R. With this in mind, one can show that R is integrally closed if and only if R satisfies Serre's conditions R_1 and S_2. The proof of this is almost identical to the proof of Proposition A.

We need a special case of standard result concerning flatness before proving one of our main results. This result is known as the *local criterion for flatness*. In general, one does not have to assume that the given ring map is faithfully flat.

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Theorem D4. Let $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a faithfully local homomorphism of Noetherian local rings. A finitely generated S-module M is flat over R if and only if $\mathrm{Tor}_1^R(k, M) = 0$.

Sketch of Proof. If M is flat, then $\mathrm{Tor}_1^R(N, M) = 0$ for all R-modules by applying the long exact sequence in Tor associated to the short exact sequence $0 \to K \to F \to N \to 0$, where F is a free R-module.

Conversely, of $\mathrm{Tor}_1^R(N, M) = 0$ for all N, then M is flat since if

$$0 \to A \to B \to C \to 0,$$

is an exact sequence of R-modules, we have and exact sequence

$$\mathrm{Tor}_1^R(C, M) \to A \otimes M \to B \otimes M \to C \otimes M \to 0.$$

Since $\mathrm{Tor}_1^R(C, M) = 0$, the map $A \otimes M \to \beta \otimes M$ is injective, showing M is flat. We now make a series of reductions.

Step 1. M is flat over R if $\mathrm{Tor}_1^R(N, M) = 0$, for all finitely generate R-modules. This follows since $N = \varinjlim N_i$ is a direct limit of finitely generated R-modules and $\varinjlim \mathrm{Tor}_1^R(N_i, M) = \mathrm{Tor}_1^R(\varinjlim N_i, M)$, M is flat over R if $\mathrm{Tor}_1^R(N, M) = 0$, for all finitely generated R-modules N.

Step 2. Let N be a finitely generated R-module. Then N has a filtration $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_r = N$, such that $N_i/N_{i-1} \cong R/P_i$, where each $P_i \subseteq R$ is a prime ideal. If $\mathrm{Tor}_1^R(R/P_i, M) = 0$, for all i, then induction on i together with the long exact Tor sequence applied to the sequences $0 \to N_{i-1} \to N_i \to N_i/N_{i-1} \to 0$, shows that $\mathrm{Tor}_1^R(N_i, M) = 0$ for all i and hence $\mathrm{Tor}_1^R(N, M) = 0$. Thus, it suffices to prove $\mathrm{Tor}_1^R(R/I, M) = 0$, for all ideals $I \subseteq R$.

Step 3. Suppose $\mathrm{Tor}_1^R(R/J, M) = 0$, for all $\mathfrak{m}$-primary ideals $J \subseteq R$. Let $I \subseteq R$ be an ideal. Fix $t \geq 1$. Then $J := I + \mathfrak{m}^t$ is $\mathfrak{m}$-primary. Let $0 \to K \to F \to M \to 0$ be an exact sequence with F finitely generated and free over S. Then $\mathrm{Tor}_1^R(R/J, M) = 0$ implies $JF \cap K = JK$, since the long exact Tor sequence implies $(JF \cap K)/JK = \mathrm{Tor}_1^R(R/J, M)$. Thus $IF \cap K \subseteq (I + \mathfrak{m}^t)K \subseteq (I + \mathfrak{n}^t)K$.

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Since K is finitely generated and S is local, taking this last intersection over all t shows $IF \cap K = IK$. Since F is flat over R, the long exact Tor sequence with I now shows $\mathrm{Tor}_1^R(R/I, M) = 0$. Thus, it suffices to shows $\mathrm{Tor}_1^R(R/J, M) = 0$ for all $\mathfrak{m}$-primary ideals J.

Step 4. Let $J \subseteq R$ be $\mathfrak{m}$-primary. It suffices to prove $\mathrm{Tor}_1^R(N, M) = 0$, for all finite length R-modules N. Proceeding by indeuction on the length, when the length is one, $N \cong k$, and our assumption gives $\mathrm{Tor}_1^R(N, M) = 0$. When N has length greater than one, we can find an R-module $N' \subseteq N$ such that N/N' has length one. We then apply the long exact Tor sequence associated to $0 \to N' \to N \to N/N' \to 0$ to complete the proof. □

Here is an important corollary.

Corollary E4. Let $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}.l)$ be a flat local homomorphism of local rings. Suppose $\underline{x} = x_1, \ldots, x_r \in S$ have the property that their images in $S/\mathfrak{m}S$ form a regular sequence. Then $\underline{x}$ forms a regular sequence in S and $S/(\underline{x})S$ is flat over R.

Proof. It suffice to prove the case $r = 1$. So, suppose $x \in S$ is a non-zerodivisor on $S/\mathfrak{m}S$. Take $s \in S$ and suppose $sx = 0$. Then $sx \in \mathfrak{m}S$, so $s \in \mathfrak{m}S$. Let $a_1, \ldots, a_d \in R$ be a minimal generating set for $\mathfrak{m}$. Then we have part of a minimal resolution of $\mathfrak{m}$ over R:

$$R^c \xrightarrow{\alpha} R^d \to \mathfrak{m} \to 0,$$

where the matrix α has entries in $\mathfrak{m}$. Tensoring with S, we preserve exactness and have $S^c \xrightarrow{\alpha\otimes 1} S^d \to \mathfrak{m}S \to 0$. On the other hand, we may write $s = s_1a_1 + \cdots s_da_d$, with $s_i \in S$. Therefore, $0 = (xs_1)a_1 + \cdots + (xs_d)a_d$. It follows that the column vector $\begin{pmatrix} xs_1 \\ \vdots \\ xs_d \end{pmatrix}$ belongs to the image of $\alpha \otimes 1$. Thus each $xs_i \in \mathfrak{m}S$.

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Therefore, by our assumption on x, each $s_i \in \mathfrak{m}S$. Thus, $s \in \mathfrak{m}^2 S$. Repeating the argument shows $s \in \mathfrak{m}^t S$, for all t, and thus, $s = 0$. Therefore, x is a non-zerodivisor in S.

Now, consider the exact sequence $0 \to S \xrightarrow{x} S \to S/xS \to 0$. Tensoring with k we get:

$$\cdots \to \mathrm{Tor}_1^R(k, S) \to \mathrm{Tor}_1^R(k, S/xS) \to S/\mathfrak{m}S \xrightarrow{x} S/\mathfrak{m}S \to S/(x, \mathfrak{m})S \to 0.$$

In the Tor sequence above, multiplication by x is injective, by the assumption on x, and $\mathrm{Tor}_1^R(k, S) = 0$, since S is flat over R. Therefore $\mathrm{Tor}_1^R(k, S/xS) = 0$, and thus S/xS is flat over R, as required. $\square$