SOLVING CUBICS AND QUARTICS BY RADICALS

D. Katz

The purpose of this handout is to record for my algebra class the classical formulas expressing the roots of degree three and degree four polynomials in terms of radicals. We begin with some general comments. Let F be a field and $f(X)$ a polynomial with coefficients in F. We will assume throughout that F contains the rational numbers. In searching for the roots of $f(X)$, it is convenient to simplify $f(X)$ by eliminating some of its terms. This can often be done by making a substitution of the form $X = Y + \lambda$, for some $\lambda \in F$. If we set $g(Y) := f(Y + \lambda)$, then clearly α is a root of $f(X)$ if and only if $\alpha - \lambda$ is a root of $g(Y)$. Thus, any procedure leading to the roots of $g(Y)$ leads to the roots of $f(X)$ and conversely. We will employ this technique in each of the cases below.

Cardano's formulas for the roots of a cubic polynomial. We begin with a cubic polynomial having coefficients in F, say $f(X) = X^3 + aX^2 + bX + c$. If we set $X := Y - a/3$, then for $g(Y) := f(Y - a/3)$, we obtain $g(Y) = Y^3 + pY + q$, where

$$p = \frac{1}{3}(3b - a^2) \quad \text{and} \quad q = \frac{1}{27}(2a^3 - 9ab + 27c).$$

Thus, by our comments above, we may start over assuming $f(X) = X^3 + pX + q$. We now consider the *discriminant* of $f(X)$, $D := -4p^3 - 27q^2$ and set

$$A := \sqrt[3]{\frac{-27}{2}q + \frac{3}{2}\sqrt{-3D}} \quad \text{and} \quad B := \sqrt[3]{\frac{-27}{2}q - \frac{3}{2}\sqrt{-3D}},$$

where the cube roots are chosen so that $A \cdot B = -3p$. To elaborate, if we take $\mu_1 := \frac{-27}{2}q + \frac{3}{2}\sqrt{-3D}$, then μ_1 has three cube roots : $\alpha_1, \alpha_2, \alpha_3$, where $\alpha_2 = \alpha_1\omega$, $\alpha_3 = \alpha_1\omega^2$ and ω is a primitive cube root of 1, in other words, $\omega^2 + \omega + 1 = 0$. Similarly, if we also set $\mu_2 := \frac{-27}{2}q - \frac{3}{2}\sqrt{-3D}$, then μ_2 has three cube roots, $\beta_1, \beta_2, \beta_3$ with $\beta_2 = \beta_1\omega$ and $\beta_3 = \beta_1\omega^2$. Now

$$(\alpha_1 \cdot \beta_1)^3 = -27p^3 = (-3p)^3.$$

Therefore $\alpha_1\beta_1 = -3p, -3p\omega$ or $-3p\omega^2$. If $\alpha_1\beta_1 = -3p$, we take $A = \alpha_1$ and $B = \beta_1$. If $\alpha_1\beta_1 = -3p\omega$, then $\alpha_2\beta_2 = -3p$, so we take $A = \alpha_2$ and $B = \beta_2$. If $\alpha_1\beta_1 = -3p\omega^2$, then $\alpha_2\beta_1 = -3p$ and we set $A = \alpha_2$, $B = \beta_1$. In other words, we can choose cube roots A and B of μ_1 and μ_2, so that $A \cdot B = -3p$.

We now show that the roots of $f(X)$ are :

$$\alpha = \frac{A+B}{3} \qquad \beta = \frac{\omega^2 A + \omega B}{3} \qquad \gamma = \frac{\omega A + \omega^2 B}{3}.$$

To do this, we calculate.

$$3(\alpha+\beta+\gamma) = (1+\omega+\omega^2)A + (1+\omega+\omega^2)B = 0,$$

so $-(\alpha+\beta+\gamma) = 0$. Similarly,

$$9(\alpha\beta+\alpha\gamma+\beta\gamma) = (1+\omega+\omega^2)A^2 + 3(\omega+\omega^2)AB + (1+\omega+\omega^2)B^2,$$

so, $9(\alpha\beta+\alpha\gamma+\beta\gamma) = -3AB = 9p$. Therefore $\alpha\beta+\alpha\gamma+\beta\gamma = p$. Finally,

$$27\alpha\beta\gamma = \omega^3A^3 + (\omega^2+\omega^3+\omega^4)A^2B + (\omega^2+\omega^3+\omega^4)AB^2 + \omega^3B^3,$$

so, $27\alpha\beta\gamma = A^3 + B^3 = -27q$. Therefore, $-\alpha\beta\gamma = q$. It now follows that we may factor $f(X) = (X-\alpha)(X-\beta)(X-\gamma)$, which is what we want.

Suppose $F = \mathbb{Q}$, the rational numbers. Then either $f(X)$ has one real root and two imaginary roots or $f(X)$ has three real roots. We can use the discriminant D to tell us which case occurs. Write $f(X) = (X-\alpha)(X-\beta)(X-\gamma)$. Taking the derivative gives

$$f'(\alpha) = (\alpha-\beta)(\alpha-\gamma), \quad f'(\beta) = (\beta-\alpha)(\beta-\gamma) \quad \text{and} \quad f'(\gamma) = (\gamma-\alpha)(\gamma-\beta).$$

Therefore, $-f'(\alpha)f'(\beta)f'(\gamma) = [(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)]^2$. Temporarily call this number $\tilde{D}$. We will show $\tilde{D} = D$. Since $f'(X) = 3X^2 + p$, we have

$$\begin{aligned} -\tilde{D} =& (3\alpha^2+p)(3\beta^2+p)(3\gamma^2+p) \\ =& 27\alpha^2\beta^2\gamma^2 + 9p(\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2) + 3p^2(\alpha^2+\beta^2+\gamma^2) + p^3. \end{aligned}$$

Of course, $\alpha^2\beta^2\gamma^2 = (-q)^2 = q^2$. A straight forward calculation shows that

$$\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 = (\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2(\alpha + \beta + \gamma)(\alpha\beta\gamma),$$

so, $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 = p^2$. Similarly,

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma),$$

so, $\alpha^2 + \beta^2 + \gamma^2 = -2p$. Therefore $-\tilde{D} = 4p^3 + 27q^2$, and $\tilde{D} = D$, as claimed.

Now, suppose that α is real and β, γ are imaginary, say $\beta = a + bi$ and $\gamma = a - bi$, with $a, b \in \mathbb{R}$ and $b \neq 0$. Then

$$\sqrt{D} = [(\alpha - (a + bi)(\alpha - (a - bi)((a + bi) - (a - bi))] = 2bi[(\alpha - a)^2 + b^2],$$

which is purely imaginary. Thus, $D \leq 0$. Conversely, if $D \leq 0$, in the formulas for A and B above, we may choose both to be real. It follows that α is real and β and γ are nonreal complex numbers. If all three roots are real, then taking $D = [(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)]^2$ shows that $D \geq 0$. Note that if $D = 0$, then $f(X)$ has repeated roots. Also note that if $D > 0$, then the formulas for the roots involve radicals of nonreal numbers. If $f(X)$ is irreducible over $\mathbb{Q}$, it can be shown that this is necessary.

Examples. (i) Suppose $f(X) = X^3 - X + 1$. Then $D = -4(-1)^3 - 27(1)^2 = -23$, so we expect one real root and two complex conjugate roots. Using the formulas above, we get

$$A = \sqrt[3]{\frac{-27}{2} + \frac{3}{2}\sqrt{69}} \qquad \text{and} \qquad B = \sqrt[3]{\frac{-27}{2} - \frac{3}{2}\sqrt{69}},$$

where we choose A to be the real cube root of $\frac{-27}{2} + \frac{3}{2}\sqrt{69}$. From $A \cdot B = -3p = 3$, it follows that B is also real. Thus, the root α is real and the roots β and γ are complex conjugates.

(ii) Suppose $f(X) = X^3 - 21X - 7$. Then $D = -4(-21)^3 - 27(-7)^2 = 3^6 7^2$, so $f(X)$ has three real roots. Using the formulas above, we obtain

$$A = 3\sqrt[3]{\frac{7}{2} + \frac{21}{2}\sqrt{-3}} \qquad \text{and} \qquad B = 3\sqrt[3]{\frac{7}{2} - \frac{21}{2}\sqrt{-3}},$$

so that the formulas express the roots of $f(X)$ in terms of cube roots of complex numbers. At first blush, it may not seem that the roots α, β and γ are real, but direct calculation shows that they are. For example, if $z = a + bi$ is a complex number, then we may write $z = re^{i\theta}$, where $r = \sqrt{a^2 + b^2}$ and θ is the angle determined by z and the x-axis. Letting $\sqrt[3]{r}$ denote the real cube root of r, we have that $(\sqrt[3]{r}e^{i(\theta/3)})^3 = z$. Set $\sqrt[3]{z} := \sqrt[3]{r}e^{i(\theta/3)}$. Using the fact that $e^{i\theta} = cos(\theta) + isin(\theta)$, it follows that $\sqrt[3]{z} + \sqrt[3]{\overline{z}}$ is real. Applying this to $z = \frac{7}{2} + \frac{21}{2}\sqrt{-3}$, shows that $\alpha = \frac{A+B}{3}$ is real. Similarly, one can show directly that β and γ are real as well. Note that $f(-5) = -27$, $f(-1) = 13$, $f(0) = -7$ and $f(5) = 13$, which gives an alternate way to see that $f(X)$ has three real roots.

(iii) The formulas above can also be deceiving. Let $f(x) = x^3 - 7x + 6 = (x-1)(x-2)(x+3)$. Then from Cardano's formulas one obtains

$$\frac{A+B}{3} = \sqrt[3]{\frac{1}{2}(-6 + \sqrt{\frac{-400}{27}})} + \sqrt[3]{\frac{1}{2}(-6 - \sqrt{\frac{-400}{27}})},$$

which is either 1, 2 or 3. Is it clear which one it is ?!

The quartic case. It turns out that extracting the roots from a degree 4 polynomial reduces to the degree three case. We start with an irreducible quartic polynomial $f(X) = X^4 + aX^3 + bX^2 + cX + d$ having coefficients in the field F containing $\mathbb{Q}$. As before, we can simplify by setting $X := Y - a/4$. Then for $g(Y) := f(Y - a/4)$, we obtain $g(Y) = Y^4 + pY^2 + qY + r$, where

$$p = \frac{1}{8}(-3a^2 + 8b), \quad q = \frac{1}{8}(a^3 - 4ab + 8c), \quad r = \frac{1}{256}(-3a^4 + 16a^2b - 64ac + 256d).$$

Thus, we may begin again assuming that $f(X) = X^4 + pX^2 + qX + r$, with $p, q, r \in F$. Then $f(X)$ has distinct roots, say $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Set

$$\begin{aligned} \theta_1 :=& (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) \\ \theta_2 :=& (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4) \\ \theta_3 :=& (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3) \end{aligned}$$

and $s_1 := \theta_1 + \theta_2 + \theta_3$, $s_2 := \theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3$ and $s_3 := \theta_1\theta_2\theta_3$. Using the factorization $f(X) = (X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)$, an easy calculation shows that $s_1 = 2p$, $s_2 = p^2 - 4r$ and $s_3 = -q^2$. It follows from this that θ_1, θ_2 and θ_3 are the roots of $h(X) := X^3 - 2pX^2 + (p^2 - 4r)X + q^2$, the so-called *resolvent cubic* associated to $f(X)$.

Now,

$$(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)=\theta_1 \qquad \text{and} \qquad (\alpha_1+\alpha_2)+(\alpha_3+\alpha_4)=0.$$

Therefore, we may write $\alpha_1+\alpha_2=\sqrt{-\theta_1}$ and $\alpha_3+\alpha_4=-\sqrt{-\theta_1}$. Similarly, $\alpha_1+\alpha_3=\sqrt{-\theta_2}$, $\alpha_2+\alpha_4=-\sqrt{-\theta_2}$, $\alpha_1+\alpha_4=\sqrt{-\theta_3}$ and $\alpha_2+\alpha_3=-\sqrt{-\theta_3}$. (Since $\sqrt{-\theta_1}\sqrt{-\theta_2}\sqrt{-\theta_3}=-q$, the choice of two square roots determines the third.) If we now add $(\alpha_1+\alpha_2)+(\alpha_1+\alpha_3)+(\alpha_1+\alpha_4)$, we get $2\alpha_1$ on the one hand and $\sqrt{-\theta_1}+\sqrt{-\theta_2}+\sqrt{-\theta_3}$ on the other. Doing likewise for $\alpha_2, \alpha_3, \alpha_4$, we find

$$\begin{aligned}
\alpha_1 &= \frac{1}{2}(\sqrt{-\theta_1}+\sqrt{-\theta_2}+\sqrt{-\theta_3})\\
\alpha_2 &= \frac{1}{2}(\sqrt{-\theta_1}-\sqrt{-\theta_2}-\sqrt{-\theta_3})\\
\alpha_3 &= \frac{1}{2}(-\sqrt{-\theta_1}+\sqrt{-\theta_2}-\sqrt{-\theta_3})\\
\alpha_4 &= \frac{1}{2}(-\sqrt{-\theta_1}-\sqrt{-\theta_2}+\sqrt{-\theta_3}),
\end{aligned}$$

which reduces the solution of the quartic equation to the solution of the associated resolvent cubic.

Department of Mathematics, University of Kansas, Lawrence KS, 66045