SPRING 2024: BONUS PROBLEM 5

BP 5. Let C be an $s \times t$ matrix with entries in $\mathbb{R}$. Suppose $u \in \mathbb{R}^t$ is a column vector with the following property: u is in the null space of C and u^t is in the row space of C. Prove that $u = \vec{0}$. Due at the start of class on Friday, April 19. (5 points)

Solution 1. Let $R_1, \ldots, R_s$ denote the rows of C, so that $R_1 u = \cdots = R_s u = 0$, where $R_i u$ means the row R_i times the column u. If u^t is in the row space of C, then we may write $u^t = a_1 R_1 + \cdots + a_s R_s$, where each $a_i \in \mathbb{R}$. Then

$$u^t u = (a_1 R_1 + \cdots + a_s R_s)u = a_1(R_1 u) + \cdots + a_s(R_s u) = a_1 0 + \cdots + a_s 0 = 0.$$

If $u = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_t \end{pmatrix}$, from $u^t u = 0$, we have $\alpha_1^2 + \cdots + \alpha_t^2 = 0$, which means each $\alpha_i = 0$, and therefore, $u = 0$.

Solution 2. We have $Cu = \vec{0}$, since u is in the null space of C and $u^t = v^t C$, for some $v \in \mathbb{R}^s$, since u^t is in the row space of C. Thus, $u^t u = (v^t C)u = v^t(Cu) = 0$. By the last sentence of the proof above, $u = \vec{0}$.