FALL 2019: MATH 558 QUIZ 2 SOLUTIONS

Use the First Principle of Mathematical Induction to prove:

\[ 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}, \]

for all \( n \geq 1 \). Be sure to identify the base case and the inductive hypothesis.

Solution. The base case, \( n = 1 \): \( 1 = \frac{1(1+1)(2\cdot1+1)}{6} = \frac{6}{6} = 1 \).

The inductive step: Assume

\[ 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}, \]

for \( n \geq 1 \) and use this to prove the \( n + 1 \) case. We add \( (n + 1)^2 \) to both sides of the equation above to get:

\[
1^2 + 2^2 + 3^2 + \cdots + n^2 + (n + 1)^2 = \frac{n(n + 1)(2n + 1)}{6} + (n + 1)^2 \\
= \frac{n(n + 1)(2n + 1)}{6} + \frac{(n + 1)^2}{6} \\
= \frac{n(n + 1)(2n + 1) + 6(n + 1)^2}{6} \\
= \frac{(n + 1)\{n(2n + 1) + 6(n + 1)\}}{6} \\
= \frac{(n + 1)\{2n^2 + 7n + 6\}}{6} \\
= \frac{(n + 1)(n + 2)(2n + 3)}{6} \\
= \frac{(n + 1)((n + 1) + 1)(2(n + 1) + 1)}{6}.
\]