Each question is worth 5 points.

1. Let $G$ be a group and $H$ a subset of $G$. Define what it means for $H$ to be a subgroup of $G$. Then give an example of a group $G$ with subgroup $H$ so that $H$ is a proper subgroup of $G$ i.e., $H \neq e$ and $H \neq G$.

**Solution.** $\text{SL}_2(\mathbb{R})$ is a proper subgroup of $\text{GL}_2(\mathbb{R})$. $2\mathbb{Z}$ is a proper subgroup of $\mathbb{Z}$.

2. Recall $S_3 = \{I, \tau, \tau^2, \sigma, \tau \sigma, \tau^2 \sigma\}$, where $\tau^3 = \sigma^2 = I$ and $\sigma \tau = \tau^2 \sigma$. Let $H = \{I, \tau, \tau^2\}$ and $K = \{I, \sigma\}$ be subgroups. Find:

   (i) The distinct right cosets of $H$ and the distinct right cosets of $K$, writing the elements in these sets in terms of the expressions given above.

   (ii) Find a group element $g \in S_3$ such that $gK \neq Kg$. Justify your answer.

**Solution.** For part (i): $H = \{I, \tau, \tau^2\}$ and $H\sigma = \{\sigma, \tau\sigma, \tau^2\sigma\}$ are the distinct right cosets of $H$. The distinct right cosets of $K$ are: $K = \{I, \sigma\}, K\tau = \{\tau, \sigma\tau\} = \{\tau, \tau^2\sigma\}$, and $K\tau^2 = \{\tau^2, \sigma\tau^2\} = \{\tau^2, \tau\sigma\}$.

   For part (ii): Take $g = \tau$. Then $\tau K = \{\tau, \tau\sigma\}$ and $K\tau = \{\tau, \tau^2\sigma\}$, so the left coset of $K$ with respect to $\tau$ is not equal to the right coset of $K$ with respect to $\tau$. 