Lecture 5: Homogeneous Equations and Properties of Matrices

Definition

A system of linear equations is said to be **homogeneous** if the right hand side of each equation is zero, i.e., each equation in the system has the form

$$(*) \quad a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0.$$

Note that $x_1 = x_2 = \cdots = x_n = 0$ is always a solution to a homogeneous system of equations, called the **trivial solution.**

Any other solution is a **non-trivial solution.**

Two Important Properties. 1. **Sums of solutions are solutions.**

Suppose $(s_1, \ldots, s_n)$ and $(s_1', \ldots, s_n')$ are solutions to (*). Then

$$a_1s_1 + \cdots + a_ns_n = 0$$
$$a_1s_1' + \cdots + a_ns_n' = 0$$

Adding, we get: $a_1(s_1 + s_1') + \cdots + a_n(s_n + s_n') = 0$, so that $(s_1 + s_1', \ldots, s_n + s_n')$ is also a solution.

2. **A scalar multiple of a solution to (*) is a solution.** Suppose $(s_1, \dots, s_n)$ is a solution, so that

$$(**) \quad a_1 s_1 + \cdots + a_n s_n = 0.$$

Let $\lambda \in \mathbb{R}$. By a *scalar multiple* of a solution, we mean

$$\lambda \cdot (s_1, \dots, s_n) = (\lambda \cdot s_1, \dots, \lambda s_n).$$

If we multiply (**) above by λ we get

$$a_1 \cdot (\lambda s_1) + \cdots + a_n \cdot (\lambda s_n) = 0,$$

which shows that $(\lambda s_1, \dots, \lambda s_n)$ is a solution to (*).

Important Consequence: Sums and scalar multiples of solutions to a homogenous system of linear equations are again solutions to the same system of equations.

Theorem

Suppose $v_1, \ldots, v_k \in \mathbb{R}^n$ *are solutions to a* **homogeneous** *system of* m *linear equations in* n *unknowns. Then, any linear combination* $\lambda_1 v_1 + \cdots + \lambda_k v_k$ *is also a solution.*

Moreover, given any homogenous system of m *linear equations in* n *unknowns, there exist solutions, i.e., vectors* $v_1, \ldots, v_k$, *in* $\mathbb{R}^n$ *such that* **every** *solution to the system is a linear combination of* $v_1, \ldots, v_k$.

Comments. 1. The first part of the theorem follows by combining the two Important Points from above.

2. By taking each $\lambda_j = 0$ above, one gets the zero solution, which is always a solution to any homogeneous system of linear equations.

3. It may be that the zero solution is the only solution, which is still consistent with the statement of the theorem.

4. The vectors $v_1, \ldots, v_k$ in the second paragraph are called **basic solutions**. Every solution to the system is a linear combination of the basic solutions.

Example

Suppose the augmented matrix associated to a homogenous system of equations has RREF

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0. \end{array}\right]$$

Find a set of basic solutions for the system.

Solution: The leading variables arex_1, x_3, and the free variables are x_2, x_4, x_5., which we set equal to s, t, u. We may write the general solution as the set of vectors of the form

$$\begin{bmatrix} s-2t-2u \\ s \\ -6t+u \\ t \\ u \end{bmatrix} = s \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix} + u \cdot \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

with $s, t, u \in \mathbb{R}$.

Thus, a set of basic solutions for the system of equations is:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Definition

Let A be an $m \times n$ matrix. The **rank** of A equals the number of leading 1s the RREF of A.

Important Comments concerning the rank of A.

1. The rank of A equals the rank of any matrix B obtained from A by a sequence of elementary row operations.

Why: Since A and B can both be brought to the same RREF.

2. Relations involving rank (**very important**): Suppose r equals the rank of A.

(i) r equals the number of leading variables in any consistent system of equations having A as coefficient matrix.

(ii) n equals r plus the number of free variables in any consistent system having A as coefficient matrix.

(iii) $n - r$ equals the number of basic solutions to the homogenous system of linear equations having A as its coefficient matrix.

Class Example

Give the system of equations:

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 + x_5 &= 0 \\ -x_1 - 2x_2 + 2x_3 + x_5 &= 0 \\ -x_1 - 2x_2 + 3x_3 + x_4 + 3x_5 &= 0 \end{aligned}$$

Find the rank of the coefficient matrix and a set of basic solutions.

Solution:

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 1 & 0 \\ -1 & -2 & 2 & 0 & 1 & 0 \\ -1 & -2 & 3 & 1 & 3 & 0 \end{array}\right] \xrightarrow[R_1+R_3]{R_1+R_2} \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 4 & 0 \end{array}\right]$$

$$\xrightarrow{R_2+R_1} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 4 & 0 \end{array}\right] \xrightarrow{-2\cdot R_2+R_3} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Class Example continued

Thus, the rank of the coefficient matrix equals 2. Replacing the free variables x_2, x_4, x_5 by the parameters r, s, t, we can write the solutions in vector form as

$$\begin{bmatrix} -2r-2s-3t \\ r \\ -s-2t \\ s \\ t \end{bmatrix} = r \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

The basic solutions are: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$

Definition

In order to define basic operations on matrices, we need to identify the elements in a matrix.

Thus, if A is an $m \times n$ matrix, whose entries are the real numbers a_{ij}, then we call the element located on the ith row and jth entry of A, the (i, j)-entry of A.

For example, given the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix},$$

2 is the (1,2)-entry of A, 4 is the (2,1)-entry of A, 6 is the (2,3)-entry of A.

d is the (2,2)-entry of B, e is the (3,1)-entry of B, h is the (4,2)-entry of B.

Matrix Addition

We want to see how to add matrices. There are two crucial components to matrix addition:

(i) We can only add matrices that have exactly the same dimensions.

In other words, in order words to add matrices A and B, A and B must have the same number of rows and the same number of columns.

(ii) The sum $A + B$ of matrices is obtained by adding the corresponding entries of A to the corresponding entries of B.

For example, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 10 & 12 \end{bmatrix}$, and

$C = \begin{bmatrix} -1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix}$, then we cannot add A and B, but

$$A + C = \begin{bmatrix} 1+(-1) & 2+2 & 3+(-3) \\ 4+4 & 5+(-5) & 6+6 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ 8 & 0 & 12 \end{bmatrix}.$$

Class Example

(i) Find the sum $A+B$ for $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 7 \\ 5 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & 6 \\ 3 & 3 & 7 \end{bmatrix}$.

(i) Find the sum $C+D$ for $C = \begin{bmatrix} -1 & 1 \\ 2 & -2 \\ 3 & -3 \end{bmatrix}$ and $D = \begin{bmatrix} 7 & 7 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$.

Solution:

$$A+B = \begin{bmatrix} 1+2 & 3+(-1) & 4+7 \\ 2+0 & -1+4 & 7+6 \\ 5+3 & 0+3 & 1+7 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 11 \\ 2 & 3 & 13 \\ 8 & 3 & 8 \end{bmatrix},$$

$$C+D = \begin{bmatrix} -1+7 & 1+7 \\ 2+3 & -2+1 \\ 3+4 & -3+0 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 5 & -1 \\ 7 & -3 \end{bmatrix}.$$

Scalar Multiplication of Matrices

Given any matrix A and any real number λ, we may form the new matrix $\lambda \cdot A$ by multiplying every entry of A by λ.

For example, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 1 \\ 2 & -2 \\ 3 & -3 \end{bmatrix}$, then

$$6 \cdot A = \begin{bmatrix} 6 \cdot 1 & 6 \cdot 2 & 6 \cdot 3 \\ 6 \cdot 4 & 6 \cdot 5 & 6 \cdot 6 \end{bmatrix} = \begin{bmatrix} 6 & 12 & 18 \\ 24 & 30 & 36 \end{bmatrix}$$

and

$$\pi \cdot C = \begin{bmatrix} \pi \cdot (-1) & \pi \cdot 1 \\ \pi \cdot 2 & \pi \cdot (-2) \\ \pi \cdot 3 & \pi \cdot (-3) \end{bmatrix} = \begin{bmatrix} -\pi & \pi \\ 2\pi & -2\pi \\ 3\pi & -3\pi \end{bmatrix}.$$

Class Example

For $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 7 \\ 5 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & 6 \\ 3 & 3 & 7 \end{bmatrix}$, calculate $-3 \cdot A + 5 \cdot B$.

Hint: Scalar multiply before adding.

Solution: $-3 \cdot A = \begin{bmatrix} -3 & -9 & -12 \\ -6 & 3 & -21 \\ -15 & 0 & -3 \end{bmatrix}$ and $5 \cdot B = \begin{bmatrix} 10 & -5 & 35 \\ 0 & 20 & 30 \\ 15 & 15 & 21 \end{bmatrix}$.

Thus,

$$-3 \cdot A + 5 \cdot B = \begin{bmatrix} -3 & -9 & -12 \\ -6 & 3 & -21 \\ -15 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 10 & -5 & 35 \\ 0 & 20 & 30 \\ 15 & 15 & 21 \end{bmatrix} = \begin{bmatrix} 7 & -14 & 23 \\ -6 & 23 & 9 \\ 0 & 15 & 18 \end{bmatrix}.$$

Properties of Matrix Algebra

Let A, B, C be $m \times n$ matrices, $0_{m\times n}$ the $m \times n$ matrix whose entries are all zero, $-A$ the matrix obtained by multiplying the entries of A by -1, and $\alpha, \beta \in \mathbb{R}$. Then:

(i) $A + B = B + A$

(ii) $(A + B) + C = A + (B + C)$

(iii) $\alpha \cdot (A + B) = \alpha \cdot A + \alpha \cdot B$

(iv $1 \cdot A = A$

(v) $A + 0_{m\times n} = A = 0_{m\times n} + A$

(vi) $-A + A = 0_{m\times n}$

(vii) $(\alpha + \beta) \cdot A = \alpha \cdot A + \beta \cdot A$

(viii) $(\alpha\beta) \cdot A = \alpha \cdot (\beta \cdot A)$

Class Example

For the matrices $A = \begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 0 & 7 \end{bmatrix}$, verify that

$$5 \cdot (A + B) = 5 \cdot A + 5 \cdot B \text{ and } (2+3) \cdot A = 2 \cdot A + 3 \cdot A.$$

Solution:

$$5 \cdot (A + B) = 5 \cdot \left(\begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ 0 & 7 \end{bmatrix} \right) = 5 \cdot \begin{bmatrix} 1 & 7 \\ -1 & 14 \end{bmatrix} = \begin{bmatrix} 5 & 35 \\ -5 & 70 \end{bmatrix}.$$

$$5 \cdot A + 5 \cdot B = 5 \cdot \begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix} + 5 \cdot \begin{bmatrix} -1 & 4 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ -5 & 35 \end{bmatrix} + \begin{bmatrix} -5 & 20 \\ 0 & 35 \end{bmatrix} = \begin{bmatrix} 5 & 35 \\ -5 & 70 \end{bmatrix}.$$

$$(2+3) \cdot A = 5 \cdot A = \begin{bmatrix} 10 & 15 \\ -5 & 35 \end{bmatrix}$$

$$2 \cdot A + 3 \cdot A = \begin{bmatrix} 4 & 6 \\ -2 & 14 \end{bmatrix} + \begin{bmatrix} 6 & 9 \\ -3 & 21 \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ -5 & 35 \end{bmatrix}.$$

The Transpose of a Matrix

Let A be an $m \times n$ matrix. The **transpose** of A, denoted A^t, is the $n \times m$ matrix whose rows are the columns of A. In other words A^t is obtained from A by interchanging rows and columns.

Examples: If

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}, \text{and } C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 3 & 9 & 0 \end{bmatrix},$$

then

$$A^t = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, B^t = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}, \text{and } C^t = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & 9 \\ 3 & 0 & 0 \end{bmatrix}.$$