Lecture 1: Systems of linear equations and their solutions

Course overview

Topics to be covered this semester:

- *Systems of linear equations and Gaussian elimination: Solving linear equations and applications*
- *Matrices: Arithmetic of matrices, trace and determinant of matrices*
- *Eigenvalues, eigenvectors, diagonalization and applications*
- *Orthogonality in Euclidean space*
- *Subspaces of Euclidean space, spanning sets, linearly independent sets, bases*
- *Vector spaces*
- *Subspaces of vector spaces, spanning sets and linearly independent sets in abstract vector spaces*

Something familiar: Two equations in two unknowns

Given the system of equations

$$\begin{aligned} 2x + 6y &= 8 \\ 6x - 2y &= 4 \end{aligned}$$

We have a couple of ways to solve this system.

1. We may solve for one variable in terms of the other variable. For example, from the first equation, we have $2x = 8 - 6y$, which yields $x = 4 - 3y$.

We then substitute the expression for x into the second equation to get: $6(4 - 3y) - 2y = 4$, thus, $24 - 18y - 2y = 4$, so $-20y = -20$, and thus $y = 1$.

So: $x = 4 - 3 \cdot 1 = 1$ and therefore, $x = 1$, $y = 1$ is the *unique* solution to the given system.

2. We may also solve the system by *eliminating a variable*.

For example, if we multiply the first equation by 3, we obtain

$$6x + 18y = 24$$
$$6x - 2y = 4$$

If we then subtract the second equation from the first (thereby eliminating x), we get $20y = 20$, so $y = 1$.

Substituting this into the (original) first equation yields $2x + 6 \cdot 1 = 8$, so $2x = 2$, and thus $x = 1$, as expected.

Class Example

Find the unique solution to:

$$3x - 9y = 9$$
$$6x + y = 37$$

Multiply the first equation by 2,

$$6x - 18y = 18$$
$$6x + y = 37$$

Subtracting the second equation from the first, we get: $-19y = -19$, and thus $y = 1$.

Substituting $y = 1$ into the first equation yields $3x - 9 \cdot 1 = 9$, so $3x = 18$, and $x = 6$. Therefore, $x = 6, y = 1$ is the unique solution.

We also say the ordered pair $(6, 1)$ is a solution.

Geometric Interpretation

Recall that the graph of the equation $ax + by = c$ is a straight line in the plane.

Note: If $b \neq 0$, we get the slope-intercept form $y = \frac{-a}{b} \cdot x + \frac{c}{b}$.

Thus, (u, v) is a solution to $ax + by = c$ if and only if (u, v) lies on the corresponding line.

Thus, given two equations, $ax + by = c$ and $dx + ey = f$, (u, v) is a solution to both equations if and only if lie lies on both lines.

Geometrically, there are three possibilities:

1. The lines intersect in one point
2. The lines are parallel
3. The two lines are the same

Consequences of Geometric Interpretation

It follows that a given system of equations

$$ax + by = c$$
$$dx + ey = f$$

has either

1. A unique solution (when the two lines intersect in a point) or
2. No solution (when the lines are parallel) or
3. Infinitely many solutions (when the two lines are the same)

Thus, there can never be just finitely many solutions!

For example, no system of two linear equations in two unknowns has 17 solutions.

In case 3 above, the system of two equations reduces to just one equation, say $ax + by = c$.

Suppose $a \neq 0$. Then we solve the equation for x to obtain

$$x = (-b/a)y + c/a.$$

To write the general solution, we introduce a new parameter, t, and say the solutions are

$$y = t \quad \text{and} \quad x = (-b/a)t + c/a,$$

for all real numbers t.

Alternately, we say the solutions are all ordered pairs $(\frac{-a}{b}t + \frac{c}{b}, t)$, with t any real number.

Class Example

Write the solutions to $2x + 6y = 10$ in parametric form using the parameter t.

Solution: Solving for x in terms of y, we get: $x = 5 - 3y$. Thus the solutions are:

$$x = 5 - 3t \qquad \text{and} \qquad y = t,$$

for all real numbers t or,

$$\{(5 - 3t, t) \mid t \in \mathbb{R}\}.$$

Or, solving for you in terms of x, yields $y = \frac{5}{3} - \frac{1}{3}x$, so the solutions are:

$$\{(s, \frac{5}{3} - \frac{1}{3}s) \mid s \in \mathbb{R}\}.$$

Note: we can use any parameter we like.

Linear Equations

A linear equation is an equation involving variables and coefficients, but no products or powers of variables.

Some examples:

(a) $2x + 3y = 6$

(b) $7u - 8v + \sqrt{2}y + \pi z = 17$

(c) $75x_1 + \frac{2}{19}x_2 + -23x_3 = \sqrt[3]{\pi}$

General linear equation:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \qquad (*),$$

where $a_1, \ldots, a_n, b$ are real numbers.

Solutions:

A solution to a linear equation is an ordered tuple that yields a valid equation upon substituting for the variables.

So for example, if $(s_1, \ldots, s_n)$ is an n-tuple of real numbers then $(s_1, \ldots, s_n)$ is a solution to the equation (*) if and only if

$$a_1 s_1 + \cdots + a_n s_n = b.$$

We also say that $x_1 = s_1, \ldots, x_n = s_n$ is a solution to (*).

Example

Consider the equation

$$x_1 - 2x_2 - 3x_3 = 17.$$

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The following are solutions to this equation:

(22, 1, 1), is a solution since:

$$22 - 2 \cdot 1 - 3 \cdot 1 = 17.$$

(25, -2, 4) is a solution since:

$$25 - 2(-2) - 3 \cdot 4 = 17.$$

The general solution is (17+2s+3t, s, t), where s, t are arbitrary real numbers since:

$$(17 + 2s + 3t) - 2 \cdot s - 3 \cdot t = 17.$$

Class Example

Find two particular solutions to the equation: $2x_1 - 4x_2 + 8x_3 = 6$.

(1,1,1) and (3,0,0) are two among the infinitely many solutions.To find the general solution: Solve for x_1:

$$2x_1 = 6 + 4x_2 - 8x_3,$$

so $x_1 = 3 + 2x_2 - 4x_3$. Using parameters s, t, we have that the general solution is

$$(3 + 2s - 4t, s, t).$$

Taking $s = 1, t = 1$, we get the first solution (1,1,1).

Taking $s = 0, t = 0$, we get the second solutions (3,0,0).

Systems of Linear Equations

A system of m linear equations in n unknowns is a system of equations of the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad &= \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

where all of the coefficients a_{ij} and all of the b_k are real numbers.

An ordered n-tuple $(s_1, \ldots, s_n)$, or equivalently, the set of real numbers $x_1 = s_1, \ldots, x_n = s_n$, is a *solution* to the system of equations if it is a solution to each equation in the system.

Example

Consider the system of equations

$$2x + 0y + 4z + 6w = 12$$
$$0x + 3y - 6z + 9w = 15.$$

$(6, 5, 0, 0)$ is a solution since, setting $x = 6, y = 5, z = 0, w = 0$ yields

$$2 \cdot 6 + 0 \cdot 5 + 4 \cdot 0 + 6 \cdot 0 = 12$$
$$0 \cdot 6 + 3 \cdot 5 - 6 \cdot 0 + 9 \cdot 0 = 15.$$

Class Example

Verify that, for all $s, t \in \mathbb{R}$, $(6 - 2s - 3t, 5 + 2s - 3t, s, t)$ is a solution to the system

$$\begin{aligned} 2x + 0y + 4z + 6w &= 12 \\ 0x + 3y - 6z + 9w &= 15. \end{aligned}$$

Solution: Substituing $x = 6 - 2s - 3t, y = 5 + 2s - 3t, z = s, w = t$ into the first equation yields

$$2(6-2s-3t)+0(5+2s-3t)+4s+5t = 12-4s-6t+0+4s+6t = 12,$$

while the same substitution into the second equation yields,

$$0(6-2s-3t)+3(5+2s-3t)-6s+9t = 15+6s-9t-6s+9t = 15,$$

which is what we want.