Lecture 19: Bases and Dimension Continued

Definition

Important Definition: Let $U \subseteq \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$. Vectors $v_1, \ldots, v_r \in U$ are a basis for U if:

(i) $U = \text{span}\{v_1, \ldots, v_r\}$.

(ii) The vectors $v_1, \ldots, v_r$ are linearly independent.

In particular: A basis for $\mathbb{R}^n$ is a collection of linearly independent vectors that span $\mathbb{R}^n$.

Moreover: If $v_1, \ldots, v_n$ is a basis for $\mathbb{R}^n$, then: Every vector in $\mathbb{R}^n$ can be written *uniquely* as a linear combination of $v_1, \ldots, v_n$.

Examples: (i) The standard basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is a basis for $\mathbb{R}^n$.

(ii) The basic solutions to a homogeneous system of linear equations form a basis for the solution space of that system.

(iii) If λ is an eigenvalue for the matrix A, then the basic λ-eigenvectors form a basis for E_λ, the eigenspace of λ.

Comment

Very Important Fact. Suppose the subspace U of $\mathbb{R}^n$ is spanned by the vectors $v_1, \dots, v_r$. Then there exists a subset of $v_1, \dots, v_r$ forming a basis of U.

Why: Suppose $U = \text{span}\{v_1, v_2, v_3, v_4\}$. If $v_1, \dots, v_4$ are linearly independent, they form a basis for U.

Otherwise of one the vectors is in the span of the remaining ones: say, $v_2 = av_1 + bv_3 + cv_4$.

Suppose $u \in U$. We can write

$$u = pv_1 + qv_2 + rv_3 + sv_4 = pv_1 + q(av_1 + bv_3 + cv_4) + rv_3 + sv_4$$

$$= (p + aq)v_1 + (qb + r)v_3 + (qc + s)v_4.$$

Thus, $u \in \text{span}\{v_1, v_3, v_4\}$.Thus: $U = \text{span}\{v_1, v_3, v_4\}$.

If v_1, v_3, v_4 are linearly independent, they form a basis for U. Otherwise, we may eliminate another vector and continue the process until we have a linearly independent spanning set for U, that is, a basis for U.

Example

Find a basis for the subspace $V = \text{span}\{v_1, v_2, v_3\}$, for

$$v_1 = \begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} -4 \\ -2 \\ -6 \end{bmatrix}.$$

Solution: By inspection, we see that $v_1 = 4v_2$, so that v_1 is redundant, and $V = \text{span}\{v_2, v_3\}$.

Now note that $v_3 = -2 \cdot v_2$.Thus, v_3 is redundant, and $V = \text{span}\{v_2\}$.

Thus, v_2 is a basis for V.

MAIN POINT REITERATED. Given a spanning set for a subspace, we may throw out redundant spanning vectors until we have a linearly independent spanning set – which is then a basis for that subspace.

Class Example

Find a basis for the subspace U of $\mathbb{R}^4$ spanned by the vectors

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ -5 \\ 2 \\ 5 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: We Start with the equation $A \cdot \mathbf{X} = \mathbf{0}$, with $A = [v_1 \ v_2 \ v_3]$.

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ -1 & -5 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 5 & 1 & 0 \end{array}\right] \xrightarrow[-R_1+R_4]{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \xrightarrow[-R_2+R_4]{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}\cdot R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

. The solutions to $A \cdot \mathbf{X} = \mathbf{0}$ are $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3t \\ -\frac{t}{2} \\ t \\ 0 \end{bmatrix}$.

Class Example continued

Taking $t = 1$, we have the dependence relation

$$-3v_1 - \frac{1}{2}v_2 + v_3 = \mathbf{0}. \qquad (*)$$

Thus, $v_3 = 3v_1 + \frac{3}{2}v_2$. Therefore v_3 is redundant, so v_1, v_2 span U. To see v_1, v_2 are linearly independent, suppose $v_1 = \lambda v_2$.

Then $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \lambda \cdot \begin{bmatrix} 3 \\ -2 \\ 2 \\ 5 \end{bmatrix}$. From the 3rd coordinate: $\lambda = 0$.

The first coordinate becomes $1 = 0 \cdot 3$, a contradiction. Thus, v_1, v_2 are not DEPENDENT, so they are independent. Therefore, v_1, v_2 form a basis for U.

NOTE: The same argument using (*) shows that v_1, v_3 and v_2, v_3 are also bases for U.

Fundamental Theorem

Let $v_1, \ldots, v_r$ be vectors in $\mathbb{R}^n$ that span the subspace U and suppose $w_1, \ldots, w_t \in U$ are linearly independent. Then:

(i) $t \leq r$**.** In other words:

In any given subspace, the number of linearly independent vectors is always less than or equal to the number of spanning vectors.

(ii) **Any two bases for U have the same number of elements.**
Why: If $v_1, \ldots, v_r$ and $w_1, \ldots, w_t$ are bases for U, then $t \leq r$ since the w's are linearly independent and the v's span U.

On the other hand, $r \leq t$, since the v's are linearly independent and the w's span U.

Thus, $r = t$, and the two sets of bases have the same number of elements.

Definition

Let U be a subspace of $\mathbb{R}^n$. The **dimension** of U is the number of elements in any basis of U.

Corollary. The dimension of $\mathbb{R}^n$ equals n.

WHY: $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ forms a basis for $\mathbb{R}^n$.

Corollary. The dimensions of the subspaces of $\mathbb{R}^3$ are given as follows:

(i) $\{\mathbf{0}\}$ is zero dimensional – it does not have a basis.

(ii) A line L through the origin is one dimensional. Any vector on the line forms a basis for L.

(iii) A plane P through the origin is two dimensional. Any two non-collinear vectors in P form a basis for P.

Theorem

Very Important Theorem. Let $v_1, \ldots, v_n$ be n column vectors in $\mathbb{R}^n$ and let A denote the matrix whose columns are $v_1, \ldots, v_n$. The following conditions are equivalent:

(i) A is invertible.

(ii) $\det(A) \neq 0$.

(iii) $A \cdot \mathbf{X} = \mathbf{0}$ has only the $\mathbf{0}$ solution.

(iv) $A \cdot \mathbf{X} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^n$.

(v) The vectors $v_1, \ldots, v_n$ are linearly independent.

(vi) The vectors $v_1, \ldots, v_n$ span $\mathbb{R}^n$.

(vii) The vectors $v_1, \ldots, v_n$ form a basis for $\mathbb{R}^n$.

Important Comments. (a) The equivalence of (v)-(vii) works because we are taking n vectors in $\mathbb{R}^n$. This enables us to construct an $n \times n$ matrix with the given vectors.

(b) If we take r vectors in $\mathbb{R}^n$, with $r \neq n$, then (v)-(vii) will not be equivalent.

Example

Determine if the vectors $v_1 = \begin{bmatrix} -3 \\ 2 \\ 9 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 11 \\ 19 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 39 \end{bmatrix}$ form a basis for $\mathbb{R}^3$.

Solution: The determinant of the matrix $\begin{bmatrix} -3 & 0 & 0 \\ 2 & 11 & 0 \\ 9 & 19 & 39 \end{bmatrix}$ equals $(-3) \cdot 11 \cdot 39 \neq 0$.

By the previous theorem: v_1, v_2, v_3 form a basis for $\mathbb{R}^3$.

Class Example

Which pairs of vectors in $\mathbb{R}^2$ are linearly independent:

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

and

$$w_1 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Solution: $\begin{vmatrix} 2 & 3 \\ 1 & 7 \end{vmatrix} = 11 \neq 0$, so v_1, v_2 are linearly independent.

$\begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix} = 0$, so w_1, w_2 are not linearly independent.

Example

Given the vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find a basis for $\text{span}\{v_1, v_2, v_3, v_4\}$ and determine whether or not that basis forms a basis for $\mathbb{R}^3$.

Solution: First we eliminate redundancies. Consider $A \cdot \mathbf{X} = \mathbf{0}$, for $A = [v_1 \ v_2 \ v_3 \ v_4]$.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow[-1\cdot R_2]{R_2+R_3} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[\frac{1}{2}\cdot R_3+R_2]{-R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 & 1 & 0 \end{array}\right] \xrightarrow[-R_3+R_1]{\frac{1}{2}\cdot R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 \end{array}\right].$$

Example continued

The corresponding homogenous system has non-trivial solutions, so the vectors v_1, v_2, v_3, v_4 are not linearly independent.

Since the solutions are given by $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = -t \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$, if we take $t = -2$, we see that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$ is a solution.

Thus, $v_1 + v_2 + v_3 + 2v_4 = 0$, so that $v_4 = \frac{1}{2}(v_1 + v_2 + v_3)$. Therefore, we may eliminate the redundant vector v_4.

Therefore, $\text{span}\{v_1, v_2, v_3, v_4\} = \text{span}\{v_1, v_2, v_3\}$.

Example continued

Since we now have three vectors in $\mathbb{R}^3$, we can check their linear independence by taking the determinant of the matrix whose columns are v_1, v_2, v_3.

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1 - 1 = -2 \neq 0.$$

Thus, the three vectors v_1, v_2, v_3 are linearly independent and therefore form a basis for $\mathbb{R}^3$.

Summary of Spanning, Linear Independence, and Bases

Let $v_1, \dots, v_r, w$ be columns vectors in $\mathbb{R}^n$.Let $A = [v_1 \; v_2 \cdots \; v_r]$. Then:

(i) w belongs to $\text{span}\{v_1, \dots, v_r\}$ if and only if the system of equations $A \cdot \mathbf{X} = w$ has a solution.

(ii) If $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$ is a solution to $A \cdot \mathbf{X} = w$, then $w = \lambda_1 v_1 + \cdots + \lambda_r v_r$.

(iii) $v_1, \dots, v_r$ are linearly independent if and only if $A \cdot \mathbf{X} = \mathbf{0}$ has only the zero solution.

(iv) If $v_1, \dots, v_r$ are not linearly independent and $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$ is a non-zero solution to $A \cdot \mathbf{X} = \mathbf{0}$, then

$$(*) \quad \lambda_1 v_1 + \cdots \lambda_r v_r = \mathbf{0}.$$

This means the vectors $v_1, \dots, v_r$ are linearly dependent, and thus redundant.

Summary of Spanning, Linear Independence, and Bases

(v) One can use (*) to write some v_i in terms of the remaining v's. Upon doing so:

$$\text{span}\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r\} = \text{span}\{v_1, \ldots, v_r\}.$$

(vi) One may continue to eliminate redundant vectors from among the v_i's. As soon as one one arrives at a linearly independent subset of $v_1, \ldots, v_r$, this set of vectors forms a basis for the original subspace $\text{span}\{v_1, \ldots, v_r\}$. The number of elements in the basis is then the

dimension of $\text{span}\{v_1, \ldots, v_r\}$.

(vii) To test if n vectors in $\mathbb{R}^n$ are linearly independent, or span $\mathbb{R}^n$ or form a basis for $\mathbb{R}^n$, it suffices to show that $\det[v_1 \ v_2 \ \cdots \ v_n] \neq 0$.