Lecture 13: Applications of Diagonalization

When is a matrix diagonalizable?

Theorem. Let A be an $n \times n$ matrix. The following conditions are equivalent.

(i) A is diagonalizable

(ii) $c_A(x) = (x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2} \cdots (x - \lambda_r)^{m_r}$ and for each λ_i, A has m_i basic vectors.

Moreover: When this is the case, if $v_1, \ldots, v_n$ are the n basic vectors from (ii), and we let P denote the $n \times n$ matrix whose columns are the v_i, then $P^{-1}AP$ is the $n \times n$ matrix with

$$\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_r, \ldots, \lambda_r$$

down its main diagonal, where each λ_i appears m_i times.

To summarize: The $n \times n$ matrix A is diagonalizable, if A has n eigenvalues (counted with multiplicities) and for each eigenvalue λ, if the multiplicity of λ is m, then A must have m basic eigenvectors.

Very Important Corollary. If A has n **distinct** eigenvalues, then A is diagonalizable.

Comment

Computing powers of a diagonalizable matrix: Suppose A is diagonalizable. We want to compute A^n, all n. Then $P^{-1}AP = D$, where $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Note that $D^r = \text{diag}(\lambda_1^r, \ldots, \lambda_n^r)$, for all r.

To compute the powers of A, we note that $A = PDP^{-1}$.

(i) $A^2 = PDP^{-1} \cdot PDP^{-1} = PD^2P^{-1}$.

(ii) $A^3 = A^2 \cdot A = PD^2P^{-1} \cdot PDP^{-1} = PD^3P^{-1}$.

(iii) Continuing, $A^n = PD^nP^{-1}$, for all n.

Thus, if A is diagonalizable, in order to calculate the powers of A, we just have to diagonalize A and compute the powers of a diagonal matrix.

Applications

First Application: Solving recurrence relations.

The sequence of non-negative numbers $a_0, a_1, a_2, \ldots, a_k, \ldots,$ is called a linear recursion sequence of length two if there are fixed integers α, β, c, d such that:

(i) $a_0 = \alpha$.

(ii) $a_1 = \beta$.

(iii) $a_{k+2} = c \cdot a_k + d \cdot a_{k+1}$, for all $k \geq 0$.

The conditions in (i) and (ii) are called *initial conditions.*

To solve the recurrence relation, we set up a matrix equation. Let $v_k = \begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix}$, and $A = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}$. Thus, for $k \geq 0$,

$$A \cdot v_k = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} a_{k+1} \\ a_{k+2} \end{bmatrix} = v_{k+1}.$$

Since $v_1 = Av_0$ and $v_2 = Av_1$, we have $v_2 = A^2 v_0$. And: $v_3 = Av_2 = A \cdot A^2 v = A^3 v_0$. Continuing, we have $v_k = A^k v_0$, for all k.

Applications Continued

Thus: To find a_k, we must find v_k.To find v_k, we must calculate A^k. When A is diagonalizable, this task is made easier.

We can write $A = PDP^{-1}$, with $D = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ diagonal.

Then for all $k \geq 0$, $v_k = PD^k P^{-1} \cdot v_0$.

a_k is then the first coordinate of the vector

$$PD^k P^{-1} \cdot v_0 = P \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \cdot P^{-1} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Example

The Fibonacci sequence.

$$a_0 = 0, a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3, a_6 = 5, a_7 = 8, a_8 = 13, ...$$

In general, $a_{k+2} = a_k + a_{k+1}$, for all $k \geq 0$.

To solve for a_k, proceeding as above, we write $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, and

$$v_k = \begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix} = A^k \cdot v_0 = A^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It is easy to check that $c_A(x) = x^2 - x - 1$, and thus, the eigenvalues of A are: $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$.

Since A has distinct eigenvalues, it is diagonalizable.

Example continued

To find the matrix P, we have to find the basic eigenvectors for λ_1 and λ_2.

$$\lambda_1 I_2 - A = \begin{bmatrix} \lambda_1 & -1 \\ -1 & \lambda_1 - 1 \end{bmatrix} \xrightarrow{(\lambda_1 - 1)\cdot R_1 + R_2} \begin{bmatrix} \lambda_1 & -1 \\ 0 & 0 \end{bmatrix},$$

which shows that $\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ is a basic eigenvector for λ_1. A similar calculation shows that $\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$ is a basic eigenvector for λ_2.

Thus, we take $P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}$. $P^{-1} = -\frac{1}{\sqrt{5}} \cdot \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}$.

Example continued

To calculate a_k we just need the top entry of $A^k \cdot v_0 = A^k \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We have

$$A^k \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \cdot -\frac{1}{\sqrt{5}} \cdot \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{\sqrt{5}} \cdot \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\frac{1}{\sqrt{5}} \cdot \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \cdot \begin{bmatrix} -\lambda_1^k \\ \lambda_2^k \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} \lambda_1^k - \lambda_2^k \\ - \end{bmatrix}.$$

Thus

$$a_k = \frac{\lambda_1^k - \lambda_2^k}{\sqrt{5}} = \frac{1}{\sqrt{5}} \cdot \{(\frac{1+\sqrt{5}}{2})^k - (\frac{1-\sqrt{5}}{2})^k\}$$

for all $k \geq 0$. (!!)

Applications Continued

Calculating e^A for A diagonalizable.

Suppose A is diagonalizable. Then $A = PDP^{-1}$ for D an $n \times n$ diagonal matrix with the eigenvalues of A down its main diagonal.

Thus, $A^n = PD^nP^{-1}$, for all n, as before. Therefore:

$$\begin{aligned}
e^A &= I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \\
&= I_n + (PDP^{-1}) + \frac{1}{2!}(PD^2P^{-1}) + \frac{1}{3!}(PD^3P^{-1}) + \cdots \\
&= P\{I_n + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots\}P^{-1} \\
&= Pe^DP^{-1}.
\end{aligned}$$

Applications Continued

To calculate e^D, suppose $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Then

$$\frac{1}{r!} D^r = \mathrm{diag}(\frac{\lambda_1^r}{r!}, \ldots, \frac{\lambda_n^r}{r!}).$$

Summing from r 0 to ∞, we see

$$e^D = \sum_{r=0}^{\infty} \frac{1}{r!} D^r = \sum_{r=0}^{\infty} \mathrm{diag}(\frac{\lambda_1^r}{r!}, \ldots, \frac{\lambda_n^r}{r!}) = \text{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n}).$$

For example: if $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, then we have seen that $A = PDP^{-1}$, for $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus:

$$e^A = Pe^D P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-1} & 0 \\ 0 & e \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{-1} & e \\ 0 & e \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-1} & e - e^{-1} \\ 0 & e \end{bmatrix}.$$

Class Example

Calculate e^A for the matrix $A = \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix}$. Use the fact that the eigenvalues of A are 2 and 3, $P = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$ is the diagonalizing matrix, and $P^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$.

Solution: $e^A = Pe^D P^{-1} = P \cdot \begin{bmatrix} e^2 & 0 \\ 0 & e^3 \end{bmatrix} \cdot P^{-1} =$

$$\begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} e^2 & 0 \\ 0 & e^3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 3e^2 & 2e^3 \\ -e^2 & -e^3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3e^2 - 2e^3 & 6e^2 - 6e^3 \\ -e^2 + e^3 & -2e^2 + 3e^3 \end{bmatrix}.$$

Vector Valued First Order Linear Differential Equations

Let $\mathbf{X}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ be a vector valued function of t, i.e., each component $x_i(t)$ is a function of t. The derivative of $\mathbf{X}(t)$ is just the vector valued function $\mathbf{X}'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$.

A vector valued first order linear differential equation is a vector equation of the form:

$$\mathbf{X}(t) = A \cdot \mathbf{X}'(t),$$

where A is an $n \times n$ matrix with entries in $\mathbb{R}$. The fixed vector $\mathbf{X}(0)$ is called the *initial condition*.

Vector Valued First Order Linear Differential Equations

Note that if we let $A = (a_{ij})$, then the matrix equation above is the same as the system of first order linear differential equations:

$$\begin{aligned} x_1'(t) &= a_{11}x_1(t) + \cdots + a_{1n}x_n(t) \\ x_2'(t) &= a_{21}x_1(t) + \cdots + a_{2n}x_n(t) \\ \vdots \quad &= \qquad\qquad \vdots \\ x_n'(t) &= a_{n1}x_1(t) + \cdots + a_{nn}x_n(t) \end{aligned}$$

GOAL: Solve a system of first order linear differential equations by converting to a vector valued first order linear differential equation.

If the coefficient matrix A is diagonalizable, we can solve the system.

A single first order linear differential equation: The 1×1 case

Recall from Calculus I: If $x(t) = Ce^{at}$, then $x'(t) = aCe^{at} = a \cdot x(t)$.

In other words, $x(t) = Ce^{at}$ is the **general solution** to the first order linear differential equation $x'(t) = ax(t)$.

Note that $x(0) = C$, so C is the initial condition.

Thus the solution to the differential equation $x'(t) = a \cdot x(t)$, with initial condition $x(0)$ is:

$$x(t) = x(0)e^{at}.$$

The 2×2 case

We start with the system of differential equations:

$$x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t)$$
$$x_2'(t) = a_{21}x_1(t) + a_{21}x_2(t).$$

This is equivalent to the vector equation: $\mathbf{X}'(t) = A \cdot \mathbf{X}(t)$, where $A = (a_{ij})$ is the 2×2 coefficient matrix.

Assume A is diagonalizable, so $A = PDP^{-1}$, where P is the matrix of basic eigenvectors and $D = \mathrm{diag}(\lambda_1, \lambda_2)$, where λ_1, λ_2 are the eigenvalues of A.

Set $\mathbf{Y}(t) = P^{-1} \cdot \mathbf{X}(t)$. Then $\mathbf{Y}'(t) = P^{-1} \cdot \mathbf{X}'(t)$. This leads to:

$$\begin{aligned}
\mathbf{X}'(t) &= A \cdot \mathbf{X}(t) \\
\mathbf{X}'(t) &= PD(P^{-1} \cdot \mathbf{X}(t)) \\
P^{-1} \cdot \mathbf{X}'(t) &= D(P^{-1} \cdot \mathbf{X}(t)) \\
\mathbf{Y}'(t) &= D \cdot \mathbf{Y}(t).
\end{aligned}$$

The 2×2 case

Translating the last vector equation into a system:

$$y_1'(t) = \lambda_1 y(t) \quad \text{and} \quad y_2'(t) = \lambda_2 y_2(t).$$

In other words we now have two separate, independent equations. Thus

$$y_1(t) = y_1(0)e^{\lambda_1 t} \quad \text{and} \quad y_2(t) = y_2(0)e^{\lambda_2 t}.$$

Converting back to a matrix equation, we have

$$\mathbf{Y}(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \cdot \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = e^{Dt} \cdot \mathbf{Y}(0).$$

The 2×2 case

Converting back to $\mathbf{X}$ we have:

$$P^{-1} \cdot \mathbf{X}(t) = e^{Dt} \cdot P^{-1}\mathbf{X}(0), \quad \text{and thus} \quad \mathbf{X}(t) = Pe^{Dt}P^{-1}\mathbf{X}(0).$$

Since $e^{A}t = Pe^{Dt}P^{-1}$,

$$\mathbf{X}(t) = e^{At} \cdot \mathbf{X}(0).$$

This looks just like the answer in the 1×1 case. .

This explains why we want to consider expressions e^{A}.

The solution in the $n \times n$ case takes exactly the same form.

Example

Find the solution to the system of first order linear differential equations:

$$x_1'(t) = x_2(t)$$
$$x_2'(t) = x_1(t).$$

with initial conditions: $x_1(0) = 1, x_2(0) = -1$.

Solution: The vector equation is $\mathbf{X}'(t) = A \cdot \mathbf{X}(t)$, with $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Thus, the solution is: $\mathbf{X}(t) = e^{At} \cdot \mathbf{X}(0)$, where $\mathbf{X}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The usual calculation shows that A has eigenvalues 1 and -1 with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.Thus, we take $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Then $P^{-1} = -\frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$. We aso have $Dt = \begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix}$.

Example continued

Now,

$$e^{At} = Pe^{Dt}P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \cdot -\frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} -e^t - e^{-t} & -e^t + e^{-t} \\ -e^t + e^{-t} & -e^t - e^{-t} \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathbf{X}(t) = e^{At} \cdot \mathbf{X}(0) = -\frac{1}{2} \begin{bmatrix} -e^t - e^{-t} & -e^t + e^{-t} \\ -e^t + e^{-t} & -e^t - e^{-t} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} -e^t - e^{-t} + e^t - e^{-t} \\ -e^t + e^{-t} + e^t + e^{-t} \end{bmatrix} = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$$

Thus, the solution to the system is: $x_1(t) = e^{-t}$ and $x_2(t) = -e^{-t}$.

Example

Find the solution to the system of first order linear differential equations:

$$x_1'(t) = -6x_2(t)$$
$$x_2'(t) = -x_1(t) + xy_2(t),$$

with initial conditions $x_1(0) = \sqrt{2}$ and $x_2(0) = \pi$.

Solution: In terms of matrices, we have $X'(t) = A \cdot \mathbf{X}(t)$, with $A = \begin{bmatrix} 0 & -6 \\ -1 & 5 \end{bmatrix}$ and $\mathbf{X}(0) = \begin{bmatrix} \sqrt{2} \\ \pi \end{bmatrix}$. The solution is: $e^{At} \cdot \mathbf{X}(0)$. From above, we have $e^{At} = \begin{bmatrix} 3e^{2t} - 2e^{3t} & 6e^{2t} - 6e^{3t} \\ -e^{2t} + e^{3t} & -2e^{2t} + 3e^{3t} \end{bmatrix}$ - just repeat the calculation above with the eigenvalues 2,3 replaced by $2t, 3t$.

Example continued

Thus:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathbf{X}(t) = e^{At} \cdot \mathbf{X}(0) = \begin{bmatrix} 3e^{2t} - 2e^{3t} & 6e^{2t} - 6e^{3t} \\ -e^{2t} + e^{3t} & -2e^{2t} + 3e^{3t} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2} \\ \pi \end{bmatrix}$$

$$= \begin{bmatrix} 3\sqrt{2}e^{2t} - 2\sqrt{2}e^{3t} + 6\pi e^{2t} - 6\pi e^{3t} \\ -\sqrt{2}e^{2t} + \sqrt{2}e^{3t} - 2\pi e^{2t} + 3\pi e^{3t} \end{bmatrix} = \begin{bmatrix} (3\sqrt{2} + 6\pi)e^{2t} - (2\sqrt{2} + 6\pi)e^{3t} \\ -(\sqrt{2} + 2\pi)e^{2t} + (\sqrt{2} + 3\pi)e^{3t} \end{bmatrix}.$$

Therefore the solution is :

$$x_1(t) = (3\sqrt{2} + 6\pi)e^{2t} - (2\sqrt{2} + 6\pi)e^{3t}$$
$$x_2(t) = -(\sqrt{2} + 2\pi)e^{2t} + (\sqrt{2} + 3\pi)e^{3t}.$$