Solutions to Exam 2 Practice Problems

1. $C_A(x) = \det \begin{bmatrix} x-2 & 2 & -1 \\ +1 & x-3 & 1 \\ -2 & 4 & x-3 \end{bmatrix}$

multiplicity 2

$= x^3 - 8x^2 + 13x - 6 = (x-1)^2 \cdot (x-6) \Rightarrow \lambda_1 = 1, \lambda_2 = 6$

To find E_1: $1 \cdot I_3 - A = \begin{bmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ -2 & 4 & -2 \end{bmatrix} \xrightarrow[-2R_1+R_2]{R_1+R_2} \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Basis for Eigenspace: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} +2s - t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} +2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ dimension $E_1 = 2 =$ multiplicity of λ_1

To find E_6: $6 \cdot I_3 - A = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 3 & 1 \\ -2 & 4 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & -1 \\ -2 & 4 & 3 \end{bmatrix}$

$\xrightarrow[2R_1+R_3]{-4R_1+R_2} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -10 & -5 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow$ ETC $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} =$ basis for E_6

$\dim E_6 = 1 =$ multiplicity of λ_2

$\therefore$ A is diagonalizable

$P = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ is the diagonalizing matrix

and $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

Note: P Need not be unique but its columns must come from bases of eigenspaces

$$c_B(x) = \det \begin{bmatrix} x-1 & 0 \\ -1 & x-1 \end{bmatrix} = (x-1)^2. \quad \therefore \lambda = 1$$

is the only eigenvalue - it has multiplicity 2.

For E_1: $\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \Rightarrow$ basis for E_1 is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

dimension $E_1 = 1 <$ multiplicity of $\lambda \Rightarrow$ B is

NOT diagonalizable

$$c_C(x) = \det \begin{bmatrix} x-2 & -1 & 0 \\ 0 & x-2 & -1 \\ 0 & 0 & x+1 \end{bmatrix} = (x-2)^2(x+1).$$

$\lambda_1 = 2$; $\lambda_2 = 1$ are eigenvalues. 2 has multiplicity two

For E_2: $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

is a basic sol$^{\underline{n}}$ $\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ = basic eigen vector $\Rightarrow$ dimension $E_2 = 1$

$\therefore$ C is NOT diagonalizable.

$$c_D(x) = \det \begin{bmatrix} x-1 & -2 & -3 \\ 0 & x+1 & -3 \\ 0 & 0 & x-2 \end{bmatrix} = (x-1)(x+1)(x-2)$$

$\therefore$ Eigenvalues are $1, -1, 2 \Rightarrow$ D is diagonalizable

E_1: $\begin{bmatrix} 0 & -2 & -3 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ basic eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

dimension $= 1$
$=$ multiplicity of 1

E_{-1}: $\begin{bmatrix} -2 & -2 & -3 \\ 0 & 0 & -3 \\ 0 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

basic eigenvector is $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

dimension $= 1 =$
multiplicity of -1

E_2: $\begin{bmatrix} 1 & -2 & -3 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

basic eigenvector is: $\begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow$ dimension $E_2 = 1 =$ multiplicity of E_2

Take $P = \begin{bmatrix} 1 & 1 & 5 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ for the diagonalizing matrix

2. $c_A(x) = \det\begin{bmatrix} x-1 & -4 \\ -2 & x+6 \end{bmatrix} = x^2+5x-14$
$= (x-2)(x+7)$

$\therefore$ Eigenvalues are $\lambda_1 = 2$, $\lambda_2 = -7$

$\therefore$ A is diagonalizable. Check

that $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is an eigenvector for 2

$\begin{pmatrix} +1 \\ -2 \end{pmatrix}$ is an eigen vector for -7

$\therefore P = \begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}$ is a diagonalizing matrix

$P^{-1} = -\frac{1}{9}\begin{pmatrix} -2 & -1 \\ -1 & 4 \end{pmatrix}$ $\quad D = \begin{pmatrix} 2 & 0 \\ 0 & -7 \end{pmatrix} \Rightarrow e^D = \begin{pmatrix} e^2 & 0 \\ 0 & e^{-7} \end{pmatrix}$

$P^{-1}AP = D \Rightarrow A = PDP^{-1} \Rightarrow e^A = Pe^DP^{-1}$

$= \begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} e^2 & 0 \\ 0 & e^{-7} \end{pmatrix}\begin{pmatrix} -2 & -1 \\ -1 & -4 \end{pmatrix} \cdot \left(-\frac{1}{9}\right)$

$= \begin{pmatrix} 4e^2 & e^{-7} \\ e^2 & -2e^{-7} \end{pmatrix}\begin{pmatrix} -2 & -1 \\ -1 & -4 \end{pmatrix}\left(-\frac{1}{9}\right)$

$= -\frac{1}{9}\begin{pmatrix} -8e^2 - e^{-7} & -4e^2 - 4e^{-7} \\ -2e^2 + 2e^{-7} & -e^2 + 8e^{-7} \end{pmatrix}$

In Lecture 14 we calculated e^{Bt}

(though in Lecture 14, we wrote $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$)

so $e^{Bt}\Big|_{t=1} = e^{B}$ ... so just set $t=1$

in the corresponding matrix to get e^{B}

For A^n: $A^n = P D^n P^{-1}$

$$= \begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & (-7)^n \end{pmatrix} \cdot \left(\frac{-1}{9}\right) \begin{pmatrix} -2 & -1 \\ -1 & -4 \end{pmatrix}$$

$$= -\frac{1}{9} \cdot \begin{pmatrix} 4 \cdot 2^n & (-7)^n \\ 2^n & -2 \cdot (-7)^n \end{pmatrix} \begin{pmatrix} -2 & -1 \\ -1 & -4 \end{pmatrix}$$

$$= -\frac{1}{9} \cdot \begin{pmatrix} -8 \cdot 2^n - (-7)^n & -4 \cdot 2^n - 4(-7)^n \\ -2^{n+1} + 2(-7)^n & -2^n + 8(-7)^n \end{pmatrix}$$

For B^n: $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$ $\quad P^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$$D^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3^n \end{bmatrix} \Rightarrow B^n = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3^n \\ 0 & 0 & 3^n \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^n & 3^n & 3^n \\ 3^n & 3^n & 3^n \\ 3^n & 3^n & 3^n \end{bmatrix}$$

#3) Set $v_0 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$

Then $\begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix} = A^k \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ all k

We must diagonalize A to calculate A^k.

$$C_A(x) = \det \begin{bmatrix} x & -1 \\ 2 & x-3 \end{bmatrix} = x^2 - 3x + 2 = (x-1)(x-2)$$

$\therefore$ Eigenvalues $= 1, 2$.

For E_1: $\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ = basic eigenvector

For E_2: $\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ = basic eigenvector

$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ diagonalizes A

$P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ $\quad \therefore A^k = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2^k \\ 1 & 2^{k+1} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{matrix} 2-2^k & -1+2^k \\ 2-2^{k+1} & -1+2^{k+1} \end{matrix}$$

$\therefore \begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 2-2^k & -1+2^k \\ 2-2^{k+1} & -1+2^{k+1} \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4-2^{k+1} - 4 + 4 \cdot 2^k \\ - \end{bmatrix} \Rightarrow$

$$a_k = 4\cdot 2^k - 2^{k+1} = 2\cdot 2^{k+1} - 2^{k+1} = 2^{k+1}$$

*4. $\vec{x}\,'(t) = A\cdot\vec{x}(t)$

where $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$ and $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

Sol$^{\underline{n}}$ is: $e^{At}\cdot\vec{x}(0)$, where $\vec{x}(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

To Diagonalize A: $c_A(x) = \det\begin{bmatrix} x-6 & 1 \\ -2 & x-3 \end{bmatrix} = (x-6)(x-3)+2$

$= x^2 - 9x + 20 = (x-4)(x-5) \Rightarrow \lambda_1 = 4, \lambda_2 = 5$ are the eigenvalues each with multiplicity one.

Standard Calculation $\Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ = basic eigen vector for 4

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ = Basic Eigen Vector for 5.

$P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ diagonalizes A & $D = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$

$P^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$ $\qquad e^{Dt} = \begin{pmatrix} e^{4t} & 0 \\ 0 & e^{5t} \end{pmatrix}$

$$e^{At} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} e^{4t} & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} e^{4t} & e^{5t} \\ 2e^{4t} & e^{5t} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -e^{4t}+2e^{5t} & e^{4t}-e^{5t} \\ -2e^{4t}+2e^{5t} & 2e^{4t}-e^{5t} \end{pmatrix}$$

$$\vec{X}(t) = e^{At} \cdot \vec{X}(0)$$

$$= \begin{pmatrix} -e^{4t}+2e^{5t} & e^{4t}-e^{5t} \\ -2e^{4t}+2e^{5t} & 2e^{4t}-e^{5t} \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= \begin{bmatrix} e^{4t}-2e^{5t} + 2e^{4t}-2e^{5t} \\ 2e^{4t}-2e^{5t} + 4e^{4t}-2e^{5t} \end{bmatrix} = \begin{bmatrix} 3e^{4t}-4e^{5t} \\ 6e^{4t}-4e^{5t} \end{bmatrix}$$

$\therefore$ $x_1(t) = 3e^{4t} - 4e^{5t}$

$x_2(t) = 6e^{4t} - 4e^{5t}$

5. w is in the Span of v_1, v_2, v_3 if we have a Sol^n to $A\vec{x} = w$, where

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$ Via Gaussian Elimination:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 1 & -1 & 0 & 4 \\ 0 & 1 & -1 & 8 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 1 & -1 & 8 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 10 \end{array}\right]$$

The system has NO sol^n so $w \notin$ Span$\{v_1, v_2, v_3\}$

6. The 3 vectors in $\mathbb{R}^3$ are linearly independent if $\det A \neq 0$. But $\det A = 0$ (check this)

$\Rightarrow v_1, v_2, v_3$ are NOT Linearly Independent

#7. If v_1, v_2, v_3 are L.I. they form a basis for Span$\{v_1, v_2, v_3\}$.

Check $A\bar{X} = 0$, for $A = \begin{bmatrix} 1 & -2 & 6 \\ -1 & 1 & -4 \\ 0 & 1 & -2 \\ 1 & 1 & 0 \end{bmatrix}$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 6 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & -6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 6 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \text{1 free Variable}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So there is a non-trival Sol$^{\underline{n}}$ $\Rightarrow$ NOT L.I.

Basic Sol$^{\underline{n}}$ = $\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$

$\therefore -2v_1 + 2v_2 + v_3 = \vec{0} \Rightarrow v_3 \in$ Spann$\{v_1, v_2\}$

[Note: also $v_1 \in$ Span$\{v_2, v_3\}$ and $v_2 \in$ Span$\{v_1, v_3\}$]

We throw out $v_3 \Rightarrow$ Span$\{v_1, v_2, v_3\}$ = Span$\{v_1, v_2\}$

If v_1, v_2 are L.I then they form a basis for Span$\{v_1, v_2\}$ = Span$\{v_1, v_2, v_3\}$.

Suppose they were NOT LI: $\Rightarrow v_1 = \lambda v_2$

$$\Rightarrow \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2\lambda \\ \lambda \\ \lambda \\ \lambda \end{bmatrix} \qquad 3^{rd} \text{ row} \Rightarrow \lambda = 0.$$

But then 1^{st} row $\Rightarrow 1 = 0$, $\#$

$\therefore v_1, v_2$ are LI and form a basis for Span$\{v_1, v_2, v_3\}$ = Span$\{v_1, v_2\}$

8. Done in #1.