Lecture 9: Determinants

Definition

(i) Given a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the **determinant** of A is the real number $ad - bc$. We denote this number either by $\det(A)$ or $|A|$.

(ii) For square matrices of larger size, the determinant is defined by reducing to matrices of smaller sizes.

For example, here is **expansion along the first row** of a 3×3 matrix:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei - afh - bdi + bfg + cdh - ceg.$$

Important. For the 3×3 case, one should remember the expansion in terms of 2×2 matrices.

Do NOT memorize the last expression involving sums and products of three entries at a time.

Example

Calculate $\begin{vmatrix} 2 & 4 \\ 6 & 1 \end{vmatrix}$ and $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & -1 \\ 9 & 6 & 1 \end{vmatrix}$.

Solution: $\begin{vmatrix} 2 & 4 \\ 6 & 1 \end{vmatrix} = 2 \cdot 1 - 4 \cdot 6 = 2 - 24 = -22.$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & -1 \\ 9 & 6 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 4 & -1 \\ 6 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & -1 \\ 9 & 1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 0 & 4 \\ 9 & 6 \end{vmatrix}$$

$$= 1(4 \cdot 1 - (-1) \cdot 6) - 2(0 \cdot 1 - (-1) \cdot 9) + 3(0 \cdot 6 - 4 \cdot 9)$$

$$= 10 - 2 \cdot 9 + 3 \cdot (-36) = -116.$$

Comment

If $A = (a_{ij})$ is a 3×3 matrix and we let A_{1j} denote the 2×2 matrix obtained by deleting the 1st row and jth column of A, then we can write a formula for $\det(A)$ as follows:

$$\det(A) = a_{11} \cdot |A_{11}| - a_{12} \cdot |A_{12}| + a_{13} \cdot |A_{13}|.$$

Class Example

Calculate the determinant of the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 7 & 0 & -2 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution:

$$\det(A) = 1 \cdot \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 7 & -2 \\ 3 & 1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 7 & 0 \\ 3 & 1 \end{vmatrix}$$

$$= 1(0 - (-2)) - 2(7 - (-3)2) + (-1)(7 - 0)$$

$$= 2 - 26 - 7 = -31.$$

Comment

There are many other ways to calculate the determinant.

For example, here are three other ways to calculate $\det(A)$, for $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

$$\det(A) = -d \cdot \begin{vmatrix} b & c \\ h & i \end{vmatrix} + e \cdot \begin{vmatrix} a & c \\ g & i \end{vmatrix} - f \cdot \begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

$$\det(A) = c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix} - f \cdot \begin{vmatrix} a & b \\ g & h \end{vmatrix} + i \cdot \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$\det(A) = g \cdot \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \cdot \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \cdot \begin{vmatrix} a & b \\ d & e \end{vmatrix}.$$

Note that in each case, the determinant is obtained by expanding along a row or column.

Definition

Let $A = (a_{ij})$ be an $n \times n$ matrix.

(i) Set A_{ij} to be the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of A.

(ii) Set $c_{ij}(A) = (-1)^{i+j} \cdot |A_{ij}|$. This is called the (i, j) **cofactor** of A.

Suppose $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then:

(i) $A_{23} = \begin{bmatrix} a & b \\ g & h \end{bmatrix}$, $A_{31} = \begin{bmatrix} b & c \\ e & f \end{bmatrix}$, $A_{22} = \begin{bmatrix} a & c \\ g & i \end{bmatrix}$

(ii) $c_{23}(A) = (-1)^{2+3}|A_{23}| = -(ah - bg)$

(iii) $c_{31}(A) = (-1)^{3+1}|A_{31}| = bf - ce$

(iv) $c_{22}(A) = (-1)^{2+2}|A_{22}| = ai - cg$.

Class Example

For the matrix $A = \begin{bmatrix} 1 & 2 & 9 \\ 0 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$, find $c_{12}(A)$, $c_{33}(A)$, $c_{23}(A)$.

Solution:

(i) $c_{12} = (-1)^{1+2}|A_{12}| = -1 \cdot \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} = -2$

(ii) $c_{33}(A) = (-1)^{3+3}|A_{33}| = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$

(iii) $c_{23}(A) = (-1)^{2+3}|A_{23}| = -\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = -0 = 0.$

Cofactor expansions for the determinant

Theorem. Let $A = (a_{ij})$ be an $n \times n$ matrix.

(i) Expansion along the *ith* row:

$$|A| = a_{i1} \cdot c_{i1}(A) + a_{i2} \cdot c_{i2}(A) + \cdots + a_{in} \cdot c_{in}(A).$$

(ii) Expansion along the *jth* column:

$$|A| = a_{1j} \cdot c_{1j}(A) + a_{2j} \cdot c_{2j}(A) + \cdots + a_{nj} \cdot c_{nj}(A).$$

Note. The theorem above shows that in calculating any determinant, the calculation ultimately comes down to calculating 2×2 determinants, since each $c_{ij}(A)$ can be expressed in terms of $(n-2) \times (n-2)$ cofactors, and each of those cofactors can be expressed in terms of $(n-3) \times (n-3)$ cofactors, and so on.

Example

Calculate the determinant of $A = \begin{bmatrix} 1 & 2 & 0 & 8 \\ 4 & 5 & 1 & 2 \\ 0 & 2 & 0 & 3 \\ 2 & 0 & 0 & 4 \end{bmatrix}$.

Solution: Expand along the row or column with the most zeros - in this case, the third column.

$$|A| = 0 \cdot c_{13}(A) + 1 \cdot c_{23}(A) + 0 \cdot c_{33}(A) + 0 \cdot c_{43}(A) = c_{23}(A).$$

$$c_{23}(A) = (-1)^{2+3}|A_{23}| = -\begin{vmatrix} 1 & 2 & 8 \\ 0 & 2 & 3 \\ 2 & 0 & 4 \end{vmatrix}.$$

Expanding along the first column we get

$$-\{1 \cdot \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} - 0 \cdot |A_{21}| + 2 \cdot \begin{vmatrix} 2 & 8 \\ 2 & 3 \end{vmatrix}\} = -(8 + 0 + 2 \cdot (6 - 16)) = 12.$$

Properties of the Determinant

Let A be an $n \times n$ matrix.

(i) If A has a row or column of zeros, $|A| = 0$.

(ii) If two rows or two columns of A are the same, then $|A| = 0$.

(iii) If A' is obtained from A by multiplying a row (or column) of A by a number and adding it to a different row (or column), then $|A'| = |A|$.

(iv) If A' is obtained from A by interchanging two rows or two columns, then $|A'| = -|A|$.

(v) If A' is obtained from A by multiplying a row (or column) of A by $\lambda \neq 0$, then $|A'| = \lambda \cdot |A|$.

Note. If A' is obtained from A by an elementary row operation, then items (iii)-(iv) relate $|A'|$ to $|A|$.

In particular: If A' is obtained from A by a sequence of elementary row operations, then $|A'| \neq 0$ if and only if $|A| \neq 0$.

Important Theorem

Theorem. Let A be an $n \times n$ matrix. The following statements are equivalent:

(i) The RREF of A is the $n \times n$ identity matrix.

(ii) The homogenous system $A \cdot X = 0$ has a unique solution.

(iii) A is an invertible matrix.

(iv) $\det(A) \neq 0$.

WHY: We've seen the equivalence of (i)-(iii) previously.

By the comment above, the RREF of A has non-zero determinant if and only if $\det(A) \neq 0$.

But either the RREF of A has a row of zeros (in which case its determinant is zeoro) or the RREF of A is the identity.

Thus, the RREF of A is the identity if and only if $\det(A) \neq 0$.

Illustrating properties (i)-(v)

Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(i) If $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, $|A| = a \cdot 0 - b \cdot 0 = 0$. Similarly, if any other row or column consists of 0s.

(ii) If $A = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$, $|A| = ab - ba = 0$.

(iii) If $A' = \begin{bmatrix} a + \lambda c & b + \lambda d \\ c & d \end{bmatrix}$, then

$$|A'| = (a+\lambda c)d - (b+\lambda d)c = ad + \lambda cd - bc - \lambda cd = ad - bd = |A|.$$

(iv) If $A' = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$, then $|A'| = cb - da = -(ad - bc) = -|A|$.

(v) If $A' = \begin{bmatrix} a & \lambda b \\ c & \lambda d \end{bmatrix}$, $|A'| = a(\lambda d) - (b\lambda)c = \lambda(ad - bc) = \lambda|A|$.

Example

Calculate the determinant of the matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$,

$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$.

Solution:

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 1 \cdot \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 1 \cdot (24 - 0) = 24 = 1 \cdot 4 \cdot 6.$$

$$|B| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix} \xrightarrow[-3 \cdot R_1 + R_3]{-2 \cdot R_1 + R_2} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & -4 & -5 \end{vmatrix} = 0,$$

since two rows are the same.

Comment

The following property of the determinant will enable us to use elementary row operations to calculate the determinant of an $n \times n$ matrix.

Important Fact. Let A be be an $n \times n$ **upper or lower triangular matrix**. In other words, all entries below the main diagon or all entries above the main diagonal of A are zero.

Then $\det(A)$ is the product of the diagonal entries of A.

Examples : $$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 0 & 10 \end{vmatrix} = 1 \cdot 6 \cdot 2 \cdot 10 = 360$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 4 & \sqrt{2} & 0 \\ 6 & 1 & \pi \end{vmatrix} = 2 \cdot \sqrt{2} \cdot \pi$$

Example

Use elementary rows operations to evaluate the determinant of
$A = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 1 & 2 \\ 4 & 4 & 4 \end{bmatrix}$.

Solution:

$$\begin{vmatrix} 2 & 4 & 6 \\ 2 & 1 & 2 \\ 4 & 4 & 4 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 4 & 4 & 4 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 4 & 4 & 4 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 0 & -4 & -8 \end{vmatrix}$$

$$= 2 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 0 & 0 & -\frac{8}{3} \end{vmatrix} = 2 \cdot (1 \cdot (-3) \cdot (-\frac{8}{3})) = 16.$$

CHECK:

$$|A| = 2 \begin{vmatrix} 1 & 2 \\ 4 & 4 \end{vmatrix} - 4 \begin{vmatrix} 2 & 2 \\ 4 & 4 \end{vmatrix} + 6 \begin{vmatrix} 2 & 1 \\ 4 & 4 \end{vmatrix} = 2 \cdot (-4) - 4 \cdot 0 + 6 \cdot 4 = 16.$$

Class Example

Use elementary row operations to put the matrix $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & 5 \\ -1 & 1 & 2 \end{bmatrix}$ into upper triangular form, and then find $\det(A)$.

Solution:

$$\begin{vmatrix} 1 & 3 & -1 \\ 2 & 4 & 5 \\ -1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 \\ 0 & -2 & 7 \\ -1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 \\ 0 & -2 & 7 \\ 0 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 \\ 0 & -2 & 7 \\ 0 & 0 & 15 \end{vmatrix} = -30.$$

Class Example

Find the deteminant of the matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 1 \\ 3 & 6 & 9 \end{bmatrix}$.

Solution:

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 0 & 1 \\ 3 & 6 & 9 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 9 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 6 & 6 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{vmatrix} = -6.$$