Lecture 8: Matrix Inverses and Elementary Matrices

Converting from a system of equations to a matrix equation

Start with a system of m linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Let A denote the coefficient matrix and set $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$.

The we may rewrite the system of equations as a single matrix equation:

$$A \cdot \mathbf{x} = \mathbf{b}.$$

Example

For example, if the system is

$$ax + by + cz = u$$
$$dx + ey + fz = v,$$

then for $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}$, we get the matrix equation:

$$A \cdot \mathbf{x} = \mathbf{u}.$$

Comment: When we have a single numerical equation $ax = b$, i.e., one equation in one unknown, with $a \neq 0$, we can solve it by dividing both sides of the equation by a: $x = \frac{b}{a}$.

Equivalently: We multiply both sides of the equation by the multiplicative inverse of a, to get $x = a^{-1} \cdot b$. In some cases we can do this for a matrix equation.

Definition

Let A be an $n \times n$ matrix and write I_n for the $n \times n$ matrix with 1s down its main diagonal and zeroes elsewhere. I.e., $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$.

(i) We call I_n the $n \times n$ *identity matrix*. It has the property that

$$B \cdot I_n = B \text{ and } I_n \cdot C = C,$$

for all $s \times n$ matrices B and $n \times p$ matrices C.

(ii) An $n \times n$ matrix A^{-1} is called an *inverse matrix for A* if

$$A \cdot A^{-1} = I_n = A^{-1} \cdot A.$$

Thus: If $A\mathbf{x} = \mathbf{b}$ is a matrix equation, and A has an inverse, then we have:

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$$

$$(A^{-1}A)\ \mathbf{x} = A^{-1}\mathbf{b}$$

$$I_n \cdot \mathbf{x} = A^{-1}\mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Thus, if A^{-1} exists, $A^{-1}\mathbf{b}$ is the (unique) solution to the matrix equation $A\mathbf{x} = \mathbf{b}$.

Invertibility Criterion

The following theorem tells us when a matrix is invertible, and how to find its inverse.

Theorem. Let A be an $n \times n$ matrix.

(i) A has an inverse if and only if the rank of A equals n.

(ii) If A has an inverse, then the following steps lead to A^{-1}:

- (a) Form the $n \times (2n)$ augmented matrix $[A \mid I_n]$.
- (b) Perform elementary row operations until this augmented matrix has the form $[I_n \mid B]$.
- (c) The matrix B emerging on the right portion of the augmented matrix is, in fact, A^{-1}.

Very Important Point: The rank of A equals the number of leading ones in the RREF of A.

Therefore, if A is an $n \times n$ matrix, it has rank n if and only if it can be row reduced to I_n..

Thus, A has an inverse if and only if A can be row reduced to I_n.

Example

Find A^{-1}, for $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{6}\cdot R_3]{\frac{1}{4}\cdot R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{6} \end{array}\right]$$

$$\xrightarrow{-2\cdot R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{5}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{6} \end{array}\right] \xrightarrow{-\frac{1}{2}\cdot R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{12} \\ 0 & 1 & \frac{5}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{6} \end{array}\right]$$

$$\xrightarrow{-\frac{5}{4}\cdot R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{12} \\ 0 & 1 & 0 & 0 & \frac{1}{4} & -\frac{5}{24} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{6} \end{array}\right].$$

. Thus,

$$A^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{12} \\ 0 & \frac{1}{4} & -\frac{5}{24} \\ 0 & 0 & \frac{1}{6} \end{bmatrix}.$$

Important Case

The following theorem gives an easy-to-check criterion to test the invertibility of a 2×2 matrix, and an explicit formula for the inverse of a 2×2 matrix when it exists.

Theorem. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Set $\Delta = ad - bc$, the **determinant** of A. Then:

(i) A is invertible if and only if $\Delta \neq 0$.

(ii) If $\Delta \neq 0$, then $A^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

For example: Suppose $A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$. By the Theorem,

$$A^{-1} = \frac{-1}{8} \cdot \begin{bmatrix} 8 & -4 \\ -6 & 2 \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}.$$

Via Row reductions:

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 6 & 8 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{2}\cdot R_2]{\frac{1}{2}\cdot R_1} \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 3 & 4 & 0 & \frac{1}{2} \end{array}\right] \xrightarrow{-3\cdot R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & -2 & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}\cdot R_2} \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{4} & -\frac{1}{4} \end{array}\right] \xrightarrow{-2\cdot R_1 + R_1} \left[\begin{array}{cc|cc} 1 & 0 & -1 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} & -\frac{1}{4} \end{array}\right].$$

Thus, the two answers agree.

Why the Theorem works. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $\Delta = ad - bc \neq 0$. We can assume $a \neq 0$.

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{a}\cdot R_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{-c\cdot R_1 + R_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d - \frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{\Delta}{a} & -\frac{c}{a} & 1 \end{array}\right] \xrightarrow{\frac{a}{\Delta}\cdot R_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{\Delta} & \frac{a}{\Delta} \end{array}\right]$$

$$\xrightarrow{-\frac{b}{a}\cdot R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} + \frac{bc}{a\Delta} & -\frac{b}{\Delta} \\ 0 & 1 & -\frac{c}{\Delta} & \frac{a}{\Delta} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{\Delta} & -\frac{b}{\Delta} \\ 0 & 1 & -\frac{c}{\Delta} & \frac{a}{\Delta} \end{array}\right],$$

since

$$\frac{1}{a} + \frac{bc}{\Delta} = \frac{\Delta + bc}{a\Delta} = \frac{ad - bc + bc}{a\Delta} = \frac{d}{\Delta}.$$

Thus, $A^{-1} = \begin{bmatrix} \frac{d}{\Delta} & -\frac{b}{\Delta} \\ -\frac{c}{\Delta} & \frac{a}{\Delta} \end{bmatrix} = \frac{1}{\Delta} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Class Example

Find the inverse of $A = \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}$ in two ways: First, using row operations on the corresponding augmented matrix, and then using the determinant formula above. Check that the matrix you found is really the inverse.

Solution: Start with the augmented matrix: $\left[\begin{array}{cc|cc} 5 & 4 & 1 & 0 \\ 6 & 5 & 0 & 1 \end{array}\right]$, and try to transform the left hand side into I_2.

$$\left[\begin{array}{cc|cc} 5 & 4 & 1 & 0 \\ 6 & 5 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 6 & 5 & 0 & 1 \\ 5 & 4 & 1 & 0 \end{array}\right] \xrightarrow{-1 \cdot R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 5 & 4 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-5 \cdot R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 0 & -1 & 6 & -5 \end{array}\right] \xrightarrow{-1 \cdot R_2} \left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 0 & 1 & -6 & 5 \end{array}\right]$$

$$\xrightarrow{-1 \cdot R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 0 & 5 & -4 \\ 0 & 1 & -6 & 5 \end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} 5 & -4 \\ -6 & 5 \end{bmatrix}$.

Using the formula: $\Delta = 5 \cdot 5 - (-4) \cdot (-6) = 25 - 24 = 1$. Therefore,

$$A^{-1} = \frac{1}{1} \cdot \begin{bmatrix} 5 & -4 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -6 & 5 \end{bmatrix},$$

as before.

Check:

$$A \cdot A^{-1} = \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix} \cdot \begin{bmatrix} 5 & -4 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -6 & 5 \end{bmatrix} \cdot \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix} = A^{-1} \cdot A,$$

as required

Class Example

Find the inverse of the matrix $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 3 & -4 & 0 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1\cdot R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 2 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2\cdot R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & -2 & 3 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1\cdot R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 2 & -3 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 0 \\ 0 & 1 & 0 & 2 & -3 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{5}\cdot R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 0 \\ 0 & 1 & 0 & 2 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right]$$

Thus, $B^{-1} = \begin{bmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$.

Note that for the upper right hand block of $B = \begin{bmatrix} 3 & -3 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$,

$$\begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and for the lower right hand block, $5 \cdot \frac{1}{5} = 1$.

This illustrates the phenomenon that if the matrix A has the block form

$$A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix},$$

where B and C are square matrices, then A is invertible if and only if B and C are invertible, in which case

$$A^{-1} = \begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix}.$$

Properties of inverse matrices:

(i) Only a square matrix can have an inverse, i.e., A must be an $n \times n$ matrix, for some $n \geq 1$.

(ii) Not every square matrix has an inverse.

(iii) If A has an inverse, then the inverse is unique. That is, any invertible matrix has just one inverse.

(iv) If A has an inverse, then so does A^{-1}, in which case $(A^{-1})^{-1} = A$.

(v) If A and B are $n \times n$ matrices with inverses A^{-1}, B^{-1}, then $A \cdot B$ has an inverse, and $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$.

(vi) More generally, if $A_1, A_2, \ldots A_t$ are invertible square matrices of the same size, then $A_1 \cdot A_2 \cdot A_t$ is invertible and

$$(A_1 \cdot A_2 \cdots A_t)^{-1} = A_t^{-1} \cdots A_2^{-1} \cdot A_1^{-1}.$$

(vii) If A is invertible, then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Elementary Row Operations via Matrix Multiplication

We now illustrate how elementary row operations correspond to multiplication by **elementary matrices.** Fix the 2×3 matrix

$$M = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}.$$

Type 1 Elementary Matrix: Suppose we wish to interchange the rows of M. Interchange the rows of I_2 to get $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Now take the product of this matrix with M:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \end{bmatrix}.$$

If we want to change back, multiply again by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} c & d & e \\ a & b & c \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}.$$

Type 2 Elementary Matrix: Suppose we wish to multiply the second row of M by 5. Multiply the second row of I_2 by 5 to get $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$. Now take the product of this matrix with M:

$$\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} a & b & c \\ 5d & 5e & 5f \end{bmatrix}.$$

If we want to return to the original matrix M, multiply by $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$:

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ 5d & 5e & 5f \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}.$$

Type 3 Elementary Matrix: Suppose we wish to add -3 times the second row of M to the first row of M. Do this to I_2 to get $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$. Now take the product of this matrix with M:

$$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} a - 3d & b - 3e & c - 3f \\ d & e & f \end{bmatrix}.$$

If we want to return to the original matrix M, multiply by $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, which has the effect of adding 3 times the second row of the new matrix to the first:

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a - 3d & b - 3e & c - 3f \\ d & e & f \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}.$$

Definition

An $m \times m$ elementary matrix E is one of the following three types:

(i) **Type 1.** E is obtained by interchanging two rows of I_m.

(ii) **Type 2.** E is obtained from I_m by multiplying a row of I_m by a **non-zero** number.

(iii) **Type 3.** E is obtained by adding a multiple of one row of I_m to **another** row of I_n.

In other words, an $n \times n$ elementary matrix is any matrix obtained from I_m by applying an elementary row operation to I_m.

Moreover, if A is an $m \times n$ matrix, we may perform an elementary row operation on A by taking a product $E \cdot A$, where E is the corresponding elementary matrix.

In addition: If B is obtained from A by a sequence of s elementary row operations, then there are elementary matrices $E_1, \ldots, E_s$ such that:

$$B = E_s \cdots E_2 \cdot E_1 \cdot A.$$

Example

Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}$, we find a 2×2 matrix C such that $C \cdot A$ is the RREF of A.

First, use elementary row operations

$$\begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix} \xrightarrow{-3 \cdot R_1 + R_2} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \xrightarrow{\frac{1}{3} \cdot R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{-2 \cdot R_2 + R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We now take, in order, the elementary matrices used in the row reduction:

$$E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

We set $C := E_3 \cdot E_2 \cdot E_1 =$

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -1 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3 & -\frac{2}{3} \\ -1 & \frac{1}{3} \end{bmatrix}.$$

Example continued

To Check, we multiply:

$$C \cdot A = \begin{bmatrix} 3 & -\frac{2}{3} \\ -1 & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2,$$

which is the reduced row echelon form of A.

It is easy to check $A \cdot C = I_2$, so that in fact, C is the inverse of A.

Thus, the inverse of A is a product of elementary matrices.

This phenomenon is true in general.

And since the inverse of A is invertible, with inverse equal to A, we have the following important fact.

Fundamental Fact. Any invertible matrix is a product of elementary matrices.

Class Example

Write down a sequence of 3×3 elementary matrices that correspond to the following row operations on a 3×3 matrix:

(i) $7 \cdot R_3 + R_1$

(ii) $6 \cdot R_2$

(iii) $R_2 \leftrightarrow R_3$

Solution:

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Inverses of Elementary Matrices are Elementary Matrices

Let E be an $m \times m$ elementary matrix.

(i) **Type 1.** If E is obtained by interchanging the ith and jth rows of I_m, then $E^{-1} = E$.

(ii) **Type 2.** If E is obtained by multiplying the ith row of I_m by the constatnt $c \neq 0$, then E^{-1} is obtained by multiplying the ith row of I_m by $\frac{1}{c}$.

(iii) **Type 2.** If E is obtained from I_m by adding c times R_i to row R_j, then E^{-1} is obtained by adding $-c$ times R_i to R_j.

Example

Thus:

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

since

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 = \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

And

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

since

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

since

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$