Lecture 7: Matrix Equations and Matrix Inverses

Summary of Matrix Multiplication

Let A be an $m \times n$ matrix with rows $\mathbf{r}_1, \dots, \mathbf{r}_m$ and B be an $s \times p$ matrix with columns $\mathbf{c}_1, \dots, \mathbf{c}_p$. Then:

(i) We can form the product $A \cdot B$ if and only if $n = s$.

That is, we can only form the product $A \cdot B$ if the number of columns in A equals the number of rows in B.

(ii) When $r = s$, the (i, j)-entry of $A \cdot B$ is $\mathbf{r}_i \cdot \mathbf{c}_j$.

(iii) If the product matrix $A \cdot B$ exists, it is an $m \times p$ matrix.

Further Comments: Assume $A \cdot B$ exists, i.e., B is an $m \times p$ matrix.

(a) The columns of $A \cdot B$ are $A \cdot \mathbf{c}_1, A \cdot \mathbf{c}_2, \dots, A \cdot \mathbf{c}_p$.

(b) The rows of $A \cdot B$ are $\mathbf{r}_1 \cdot B, \mathbf{r}_2 \cdot B, \dots, \mathbf{r}_m \cdot B$.

(c) The product $B \cdot A$ will not exist unless $p = n$.

(d) Even if $A \cdot B$ and $B \cdot A$ are defined, $A \cdot B$ need not equal $B \cdot A$.

Properties of matrix multiplication:

Let A and B be $m \times n$ matrices, C an $n \times p$ matrix, D a $p \times t$ matrix, E an $s \times m$ matrix, and $\alpha \in \mathbb{R}$. Then:

(i) $(A \cdot C) \cdot D = A \cdot (C \cdot D)$

(ii) $(A \cdot C)^t = C^t \cdot A^t$

(iii) $(A + B) \cdot C = A \cdot C + B \cdot C$

(iv) $E \cdot (A + B) = E \cdot A + E \cdot B$

(v) $\alpha \cdot (A \cdot C) = (\alpha \cdot A) \cdot C = A \cdot (\alpha \cdot C)$

Converting from a system of equations to a matrix equation

Start with a system of m linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Let A denote the coefficient matrix and set $\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathrm{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$.

The we may rewrite the system of equations as a single matrix equation:

$$A \cdot \mathrm{x} = \mathrm{b}.$$

Example

For example, if the system is

$$\begin{aligned} ax + by + cz &= u \\ dx + ey + fz &= v, \end{aligned}$$

then for $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, $\mathrm{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathrm{u} = \begin{bmatrix} u \\ v \end{bmatrix}$, we get the matrix equation:

$$A \cdot \mathrm{x} = \mathrm{u}.$$

Comment: When we have a single numerical equation $ax = b$, i.e., one equation in one unknown, with $a \neq 0$, we can solve it by dividing both sides of the equation by a: $x = \frac{b}{a}$.

Equivalently: We multiply both sides of the equation by the multiplicative inverse of a, to get $x = a^{-1} \cdot b$. In some cases we can do this for a matrix equation.

In (extreme) detail: If we start with the simple equation $ax = b$, and multiply by a^{-1}, the sequence of steps we really have is:

$$\begin{aligned} ax &= b \\ a^{-1} \cdot (ax) &= a^{-1} \cdot b \\ (a^{-1} \cdot a)x &= a^{-1} \cdot b \\ 1 \cdot x &= a^{-1} \cdot b \\ x &= a^{-1} \cdot b \end{aligned}$$

In some cases, we can perform the same process for a matrix equation.

Definition

Let A be an $n \times n$ matrix and write I_n for the $n \times n$ matrix with 1s down its main diagonal and zeroes elsewhere. I.e., $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$.

(i) We call I_n the $n \times n$ *identity matrix*. It has the property that

$$B \cdot I_n = B \text{ and } I_n \cdot C = C,$$

for all $s \times n$ matrices B and $n \times p$ matrices C.

(ii) An $n \times n$ matrix A^{-1} is called an *inverse matrix for A* if

$$A \cdot A^{-1} = I_n = A^{-1} \cdot A.$$

Example

Consider $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$. Then :

$$A \cdot I_2 = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + 5 \cdot 0 & 3 \cdot 0 + 5 \cdot 1 \\ 1 \cdot 1 + 2 \cdot 0 & 1 \cdot 0 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$

And:

$$I_2 \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 0 \cdot 1 & 1 \cdot 5 + 0 \cdot 2 \\ 0 \cdot 3 + 1 \cdot 1 & 0 \cdot 5 + 1 \cdot 2 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$

Moreover: If we take $A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$, then:

$$A \cdot A^{-1} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = A \cdot A^{-1}.$$

Example continued

We can use A^{-1} to solve the matrix equation:

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$$

Multiplying this equation by A^{-1} we get

$$\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \cdot \left(\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$\left(\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \right) \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -22 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -22 \\ 8 \end{bmatrix}.$$

Example continued

Putting this all together, we can use an inverse matrix to solve the system of equations:

$$\begin{aligned} 3x + 5y &= 4 \\ x + 2y &= 6. \end{aligned}$$

Solution: The system corresponds to the matrix equation $A \cdot \mathrm{x} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, with $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$. From above, multiplying by A^{-1} yields

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -22 \\ 8 \end{bmatrix}.$$

In other words, the solution to the system is

$$x = -22 \text{ and } y = 8.$$

Class Example

(i) Verify that $A^{-1} = \begin{bmatrix} -4 & -7 \\ -1 & -2 \end{bmatrix}$ is the inverse of the coefficient matrix A of the system:

$$\begin{aligned} -2x + 7y &= -3 \\ x - 4y &= 6. \end{aligned}$$

(ii) Convert the system of equations to a matrix equation and use A^{-1} to first solve the matrix equation, and then find a solution to the system.

Solution: (i) $A \cdot A^{-1} =$

$$\begin{bmatrix} -2 & 7 \\ 1 & -4 \end{bmatrix} \cdot \begin{bmatrix} -4 & -7 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -2 \cdot (-4) + 7 \cdot (-1) & -2 \cdot (-7) + 7 \cdot (-2) \\ 1 \cdot (-4) + (-4) \cdot -1 & 1 \cdot (-7) + (-4) \cdot (-2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

On the other hand, $A^{-1} \cdot A =$

$$\begin{bmatrix} -4 & -7 \\ -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 7 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} -4 \cdot (-2) + (-7) \cdot 1 & -4 \cdot 7 + (-7) \cdot (-4) \\ -1 \cdot (-2) + (-2) \cdot 1 & -1 \cdot 7 + (-2) \cdot (-4) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(ii): The matrix equation is: $A \cdot \mathrm{x} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$, where $A = \begin{bmatrix} -2 & 7 \\ 1 & -4 \end{bmatrix}$ and $\mathrm{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Multiplying by A^{-1} we get

$$\mathrm{x} = I_2 \cdot \mathrm{x} = A^{-1} \cdot A \cdot \mathrm{x} = A^{-1} \begin{bmatrix} -3 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 & -7 \\ -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 6 \end{bmatrix} = \begin{bmatrix} -30 \\ -9 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} -30 \\ -9 \end{bmatrix} = \mathrm{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore, $x = -30$ and $y = -9$.

Properties of inverse matrices:

(i) Only a square matrix can have an inverse, i.e., A must be an $n \times n$ matrix, for some $n \geq 1$.

(ii) Not every square matrix has an inverse.

(iii) If A has an inverse, then the inverse is unique. That is, any invertible matrix has just one inverse.

(iv) If A has an inverse, then so does A^{-1}, in which case $(A^{-1})^{-1} = A$.

(v) If A and B are $n \times n$ matrices with inverses A^{-1}, B^{-1}, then $A \cdot B$ has an inverse, and $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$.

(vi) If A is invertible, then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Invertibility Criterion

The following theorem tells us when a matrix is invertible, and how to find its inverse.

Theorem. Let A be an $n \times n$ matrix.

(i) A has an inverse if and only if the rank of A equals n.

(ii) If A has an inverse, then the following steps lead to A^{-1}:

(a) Form the $n \times (2n)$ augmented matrix $[A \mid I_n]$.

(b) Perform elementary row operations until this augmented matrix has the form $[I_n \mid B]$.

(c) The matrix B emerging on the right portion of the augmented matrix is, in fact, A^{-1}.

Very Important Point: The rank of A equals the number of leading ones in the RREF of A.

Therefore, if A is an $n \times n$ matrix, it has rank n if and only if it can be row reduced to I_n..

Thus, if A has rank n, the left-hand side of the augmented matrix $[A \mid I_n]$ can be row reduced to I_n.

Example

Show that the matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ is invertible and find its inverse.

Solution: Start with the augmented matrix: $\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$, and try to transform the left hand side into I_2.

$$\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 3 & 5 & 1 & 0 \end{array}\right] \xrightarrow{-3 \cdot R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -3 \end{array}\right]$$

$$\xrightarrow{-1 \cdot R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 3 \end{array}\right] \xrightarrow{-2 \cdot R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -5 \\ 0 & 1 & -1 & 3 \end{array}\right].$$

Thus, A is invertible (it has rank 2) and its inverse is $A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$, as before.

Example

Determine if the matrix $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$ is invertible. If so, find B^{-1}.

Solution: $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-5 \cdot R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right]$

$$\xrightarrow{4 \cdot R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right] \xrightarrow{-2R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

$$\xrightarrow{5 \cdot R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right] \xrightarrow{-4 \cdot R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

Thus: $B^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$.

Check:

$$\begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} \cdot \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Class Example

Find the inverses of the following matrices: $A = \begin{bmatrix} 4 & -1 \\ 5 & -1 \end{bmatrix}$,

$B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, with a, b, c all non-zero.

Solution:

$$\left[\begin{array}{cc|cc} 4 & -1 & 1 & 0 \\ 5 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 5 & -1 & 0 & 1 \\ 4 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 4 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-4 \cdot R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & -1 & 5 & -4 \end{array}\right] \xrightarrow{-1 \cdot R_2} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & -5 & 4 \end{array}\right].$$

Thus, $A^{-1} = \begin{bmatrix} -1 & 1 \\ -5 & 4 \end{bmatrix}$.

Class Example continued

$$\left[\begin{array}{ccc|ccc} 3 & -4 & 0 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 2 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2 \cdot R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1 \cdot R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 2 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 0 \\ 0 & 1 & 0 & 2 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Thus, $B^{-1} = \begin{bmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B!$

Class Example continued

$$\left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 0 & b & 0 & 0 & 1 & 0 \\ 0 & 0 & c & 0 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{b}\cdot R_2]{\frac{1}{a}\cdot R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{b} & 0 \\ 0 & 0 & c & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{c}\cdot R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{b} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{c} \end{array}\right]$$

Thus, $C^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$.