Lecture 6: More Matrix Algebra

Matrix addition and scalar multiplication

Recall that we can only add matrices of the same dimension.

- If A and B are $m \times n$ matrices, then $A + B$ is the $m \times n$ matrix whose entries are obtained by adding the entries of A to the corresponding entries of B.
- If $\lambda \in \mathbb{R}$, then $\lambda \cdot A$ is the matrix obtained by multiplying every entry of A by λ.

Thus,

$$-2 \cdot \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 0 & -2 & -4 \end{bmatrix} + \pi \cdot \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 8 & 4 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} -4 & -8 & -12 \\ -16 & -20 & -24 \\ 0 & 4 & 8 \end{bmatrix} + \begin{bmatrix} \pi & -\pi & 0 \\ 0 & \pi & -\pi \\ 8\pi & 4\pi & 0 \end{bmatrix} = \begin{bmatrix} -4+\pi & -8-\pi & -12 \\ -16 & -20+\pi & -24-\pi \\ 8\pi & 4+4\pi & 8 \end{bmatrix}$$

Properties of Matrix Algebra

Let A, B, C be $m \times n$ matrices, $0_{m\times n}$ the $m \times n$ matrix whose entries are all zero, $-A$ the matrix obtained by multiplying the entries of A by -1, and $\alpha, \beta \in \mathbb{R}$. Then:

(i) $A + B = B + A$

(ii) $(A + B) + C = A + (B + C)$

(iii) $\alpha \cdot (A + B) = \alpha \cdot A + \alpha \cdot B$

(iv $1 \cdot A = A$

(v) $A + 0_{m\times n} = A = 0_{m\times n} + A$

(vi) $-A + A = 0_{m\times n}$

(vii) $(\alpha + \beta) \cdot A = \alpha \cdot A + \beta \cdot A$

(viii) $(\alpha\beta) \cdot A = \alpha \cdot (\beta \cdot A)$

The Transpose of a Matrix

Let A be an $m \times n$ matrix. The **transpose** of A, denoted A^t, is the $n \times m$ matrix whose rows are the columns of A.

In other words A^t is obtained from A by interchanging rows and columns.

Examples: If

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}, \text{and } C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 3 & 9 & 0 \end{bmatrix},$$

then

$$A^t = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, B^t = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}, \text{and } C^t = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & 9 \\ 3 & 0 & 0 \end{bmatrix}.$$

Class Example

For the matrices $A = \begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 0 & 7 \end{bmatrix}$, calculate $3 \cdot A^t + 7 \cdot B^t$.

Solution:

$$3 \cdot A^t + 7 \cdot B^t = 3 \cdot \begin{bmatrix} 2 & -1 \\ 3 & 7 \end{bmatrix} + 7 \cdot \begin{bmatrix} -1 & 0 \\ 4 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -3 \\ 9 & 21 \end{bmatrix} + \begin{bmatrix} -7 & 0 \\ 28 & 49 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 37 & 70 \end{bmatrix}.$$

Properties of the transpose:

Let A and B be $m \times n$ matrices and $\lambda \in \mathbb{R}$. Then

(i) $(A+B)^t = A^t + B^t$

(ii) $(A^t)^t = A$

(iii) $(\lambda \cdot A)^t = \lambda \cdot A^t$

For example:

$$\left(\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix} + \begin{bmatrix}-2 & -1\\ -1 & -7\end{bmatrix}\right)^t = \begin{bmatrix}-1 & 1\\ 2 & -6\end{bmatrix}^t = \begin{bmatrix}-1 & 2\\ 1 & -6\end{bmatrix}$$

and

$$\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}^t + \begin{bmatrix}-2 & -1\\ -1 & -7\end{bmatrix}^t = \begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix} + \begin{bmatrix}-2 & -1\\ -1 & -7\end{bmatrix} = \begin{bmatrix}-1 & 2\\ 1 & -6\end{bmatrix}.$$

Matrix multiplication: Step 1

Multiplying a row times a column: Let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$ be a row vector of length n and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ be a column vector of length n. The product $\mathbf{x} \cdot \mathbf{y}$ is defined by the equation

$$\mathbf{x} \cdot \mathbf{y} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Note :

- The number of elements in the row $\mathbf{x}$ must equal the number of elements in the column $\mathbf{y}$.
- This product is essentially the dot product of $\mathbf{x}$ and $\mathbf{y}$.
- The product of a $1 \times n$ with an $n \times 1$ matrix is a 1×1 matrix.

Example

Here are are some examples of multiplying a row times a column.

$$\begin{bmatrix} 1 & -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} = 1 \cdot 4 + (-2) \cdot (-2) + 4 \cdot 6 = 4 + 4 + 24 = 32$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 1 \cdot a + 2 \cdot b + 3 \cdot c + 4 \cdot d = a + 2b + 3c + 4d$$

Class Example

For $A = \begin{bmatrix} 2 & 1 & 7 \\ 3 & 0 & 8 \\ 2 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 4 & 2 \\ 7 & 7 \end{bmatrix}$, Find the product of:

(i) The second row of A times the first column of B

(ii) The third row of A times the second column of B.

For (i):

$$\begin{bmatrix} 3 & 0 & 8 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} = 3 \cdot 2 + 0 \cdot 4 + 8 \cdot 7 = 62$$

For (ii):

$$\begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 7 \end{bmatrix} = 2 \cdot (-1) + 2 \cdot 2 + 1 \cdot 7 = 9$$

Matrix Multiplication: Step 2

Multiplying a matrix times a vector: To do this, we think of the matrix A as a collection of rows. Let A be an $m \times n$ matrix with rows $\mathbf{r}_1, \ldots, \mathbf{r}_m$. As before, let $\mathbf{x}$ be a column vector of length n.

Then we define $A \cdot \mathbf{x}$ to be the column vector of length m given by the equation $A \cdot \mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \mathbf{r}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{x} \end{bmatrix}$.

More explicitly, if $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then

$$A \cdot \mathbf{x} = \begin{bmatrix} a_{11} \cdot x_1 + \cdots + a_{1n} \cdot x_n \\ a_{21} \cdot x_1 + \cdots + a_{2n} \cdot x_n \\ \vdots \\ a_{m1} \cdot x_1 + \cdots + a_{mn} \cdot x_n \end{bmatrix}.$$

Important Point

Note that A is an $m \times n$ matrix and $\mathbf{x}$ is an $n \times 1$ matrix and the product $A \cdot \mathbf{x}$ is an $m \times 1$ matrix.

Equivalently, if $\mathbf{x} \in \mathbb{R}^n$, then $A \cdot \mathbf{x} \in \mathbb{R}^m$. (**This is very important.**)

Thus, in terms of matrix dimensions

$$(m \times n) \cdot (n \times 1) = (m \times 1).$$

Some examples: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} au + bv \\ cu + dv \end{bmatrix}$,

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-1) + 3 \cdot 0 + 5 \cdot 3 + 7 \cdot 1 \\ 2 \cdot (-1) + 4 \cdot 0 + 6 \cdot 3 + 8 \cdot 1 \end{bmatrix} = \begin{bmatrix} 21 \\ 24 \end{bmatrix}.$$

Class Example

Given the matrices $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & -1 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$,

$C = \begin{bmatrix} 2 & -1 & 3 & -2 \end{bmatrix}$, and $D = \begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \end{bmatrix}$, Calculate:

(i) $C \cdot D$

(ii) $A \cdot D$

(iii) $C \cdot B$

(iv) $A \cdot B$

Class Example continued

$$C \cdot D = \begin{bmatrix} 2 & -1 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \end{bmatrix} = 2 \cdot 4 + -1 \cdot 1 + 3 \cdot 3 + -2 \cdot 2 = 12.$$

$$A \cdot D = A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & -1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 4 + 3 \cdot 1 + 4 \cdot 3 + 5 \cdot 2 \\ 1 \cdot 4 + -1 \cdot 1 + 1 \cdot 3 + -1 \cdot 2 \end{bmatrix} = \begin{bmatrix} 33 \\ 4 \end{bmatrix}$$

$$C \cdot B = \begin{bmatrix} 2 & -1 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 2 \cdot 1 + -1 \cdot 2 + 3 \cdot 3 + -2 \cdot 4 = 1.$$

Class Example

$$A \cdot B = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & -1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 3 \cdot 2 + 4 \cdot 3 + 5 \cdot 4 \\ 1 \cdot 1 + -1 \cdot 2 + 1 \cdot 3 + -1 \cdot 4 \end{bmatrix} = \begin{bmatrix} 40 \\ -2 \end{bmatrix}.$$

Matrix Multiplication: Step 3

Multiplying two matrices: Let A be an $m \times n$ matrix and B be a $n \times p$ matrix. Suppose A has rows $\mathbf{r}_1, \ldots, \mathbf{r}_m$ and B has columns $\mathbf{c}_1, \cdots, \mathbf{c}_p$,

Then $A \cdot B$ is the $m \times p$ matrix whose (i, j)-entry is $\mathbf{r}_i \cdot \mathbf{c}_j$.

Pictorially

$$A \cdot B = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{c}_1 & \mathbf{r}_1 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_1 \cdot \mathbf{c}_p \\ \mathbf{r}_2 \cdot \mathbf{c}_1 & \mathbf{r}_2 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_2 \cdot \mathbf{c}_p \\ \vdots & \vdots & \ldots & \vdots \\ \mathbf{r}_m \cdot \mathbf{c}_1 & \mathbf{r}_m \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_m \cdot \mathbf{c}_p \end{bmatrix}$$

In terms of columns (via Step 2):

$$A \cdot B = \begin{bmatrix} A \cdot \mathbf{c}_1 & A \cdot \mathbf{c}_2 & \cdots & A \cdot \mathbf{c}_p \end{bmatrix}$$

In terms of matrix dimensions:

$$(m \times n) \cdot (n \times p) = (m \times p).$$

Suppose $A = \begin{bmatrix} 2 & -1 \\ 7 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$, then

$$\mathbf{r}_1 \cdot \mathbf{c}_1 = \begin{bmatrix} 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 4, \text{the (1,1)-entry of } A \cdot B.$$

$$\mathbf{r}_2 \cdot \mathbf{c}_1 = \begin{bmatrix} 7 & 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 14, \text{the (2,1)-entry of } A \cdot B$$

$$\mathbf{r}_1 \cdot \mathbf{c}_2 = \begin{bmatrix} 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 5, \text{the (1,2)-entry of } A \cdot B$$

$$\mathbf{r}_2 \cdot \mathbf{c}_2 = \begin{bmatrix} 7 & 6 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 27, \text{the (2,2)-entry of } A \cdot B$$

Putting this all together we have

$$A \cdot B = \begin{bmatrix} 4 & 5 \\ 14 & 27 \end{bmatrix}.$$

Suppose: $B = \begin{bmatrix} 2 & 3 & 6 \\ 0 & 1 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 6 \\ 1 & 1 \\ 6 & 3 \end{bmatrix}$. We can calculate $B \cdot C$ by calculating its columns, $B \cdot \mathbf{c}_1$, $B \cdot \mathbf{c}_2$.

$$B \cdot \mathbf{c}_1 = B \cdot \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 6 \\ 0 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + 3 \cdot 1 + 6 \cdot 6 \\ 0 \cdot 3 + 1 \cdot 1 + 5 \cdot 6 \end{bmatrix} = \begin{bmatrix} 45 \\ 31 \end{bmatrix}$$

$$B \cdot \mathbf{c}_2 = B \cdot \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 6 \\ 0 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \cdot 6 + 3 \cdot 1 + 6 \cdot 3 \\ 0 \cdot 6 + 1 \cdot 1 + 5 \cdot 3 \end{bmatrix} = \begin{bmatrix} 33 \\ 16 \end{bmatrix}$$

Putting the columns together we get

$$B \cdot C = \begin{bmatrix} 45 & 33 \\ 31 & 16 \end{bmatrix}.$$

For $B = \begin{bmatrix} 2 & 3 & 6 \\ 0 & 1 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 6 \\ 1 & 1 \\ 6 & 3 \end{bmatrix}$, we can calculate the product $C \cdot B$ in the opposite order.

$$C \cdot B = \begin{bmatrix} 3 & 6 \\ 1 & 1 \\ 6 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 6 \\ 0 & 1 & 5 \end{bmatrix} =$$

$$\begin{bmatrix} 3 \cdot 2 + 6 \cdot 0 & 3 \cdot 3 + 6 \cdot 1 & 3 \cdot 6 + 6 \cdot 5 \\ 1 \cdot 2 + 1 \cdot 0 & 1 \cdot 3 + 1 \cdot 1 & 1 \cdot 6 + 1 \cdot 5 \\ 6 \cdot 2 + 3 \cdot 0 & 6 \cdot 3 + 3 \cdot 1 & 6 \cdot 6 + 3 \cdot 5 \end{bmatrix} = \begin{bmatrix} 6 & 15 & 48 \\ 2 & 4 & 11 \\ 12 & 21 & 51 \end{bmatrix}.$$

Note that $B \cdot C$ is a 2×2 matrix, while $C \cdot B$ is a 3×3 matrix.

Class Example

Let $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 & 1 \\ 2 & 1 & 0 \end{bmatrix}$. Calculate $A \cdot A$ and $A \cdot B$.

Also calculate $B^t \cdot A$ by calculating the columns $B^t \cdot \mathbf{c}_1$ and $B^t \cdot \mathbf{c}_2$, for $\mathbf{c}_1, \mathbf{c}_2$, the columns of A.

Solution:

$$A \cdot A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 + 3 \cdot 2 & 2 \cdot 3 + 3 \cdot 1 \\ 2 \cdot 2 + 1 \cdot 2 & 2 \cdot 3 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 6 & 7 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 \cdot 5 + 3 \cdot 2 & 2 \cdot 6 + 3 \cdot 1 & 2 \cdot 1 + 3 \cdot 0 \\ 2 \cdot 5 + 1 \cdot 2 & 2 \cdot 6 + 1 \cdot 1 & 2 \cdot 1 + 1 \cdot 0 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 16 & 15 & 2 \\ 12 & 13 & 2 \end{bmatrix}$$

Class Example continued

$$B^t \cdot \mathbf{c}_1 = \begin{bmatrix} 5 & 2 \\ 6 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \cdot 2 + 2 \cdot 2 \\ 6 \cdot 2 + 1 \cdot 2 \\ 1 \cdot 2 + 0 \cdot 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 14 \\ 2 \end{bmatrix}$$

$$B^t \cdot \mathbf{c}_2 = \begin{bmatrix} 5 & 2 \\ 6 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \cdot 3 + 2 \cdot 1 \\ 6 \cdot 3 + 1 \cdot 1 \\ 1 \cdot 3 + 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 17 \\ 20 \\ 3 \end{bmatrix}$$

Putting the columns together we get

$$B^t \cdot A = \begin{bmatrix} 14 & 17 \\ 14 & 20 \\ 2 & 3 \end{bmatrix}$$

Summary of Matrix Multiplication

Let A be an $m \times n$ matrix with rows $\mathbf{r}_1, \ldots, \mathbf{r}_m$ and B be an $s \times p$ matrix with columns $\mathbf{c}_1, \ldots, \mathbf{c}_p$. Then:

(i) We can form the product $A \cdot B$ if and only if $s = n$.

That is, we can only form the product $A \cdot B$ if the number of columns in A equals the number of rows in B.

(ii) When $s = n$, the (i, j)-entry of $A \cdot B$ is $\mathbf{r}_i \cdot \mathbf{c}_j$.

(iii) If the product matrix $A \cdot B$ exists, it is an $m \times p$ matrix.

Further Comments: Assume $A \cdot B$ exists, i.e., B is an $n \times p$ matrix.

(a) The columns of $A \cdot B$ are $A \cdot \mathbf{c}_1, A \cdot \mathbf{c}_2, \ldots, A \cdot \mathbf{c}_p$.

(b) The rows of $A \cdot B$ are $\mathbf{r}_1 \cdot B, \mathbf{r}_2 \cdot B, \ldots, \mathbf{r}_m \cdot B$.

(c) The product $B \cdot A$ will not exist unless $p = m$.

(d) Even if $A \cdot B$ and $B \cdot A$ are defined, $A \cdot B$ need not equal $B \cdot A$.

Properties of matrix multiplication:

Let A and B be $m \times n$ matrices, C an $n \times p$ matrix, D a $p \times t$ matrix, E an $s \times m$ matrix, and $\alpha \in \mathbb{R}$. Then:

(i) $(A \cdot C) \cdot D = A \cdot (C \cdot D)$

(ii) $(A \cdot C)^t = C^t \cdot A^t$

(iii) $(A + B) \cdot C = A \cdot C + B \cdot C$

(iv) $E \cdot (A + B) = E \cdot A + E \cdot B$

(v) $\alpha \cdot (A \cdot C) = (\alpha \cdot A) \cdot C = A \cdot (\alpha \cdot C)$

Example

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$. Then:

$$(A \cdot B)^t = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}^t = \begin{bmatrix} aw + by & cw + dy \\ ax + bz & cx + dz \end{bmatrix}$$

And

$$B^t \cdot A^t = \begin{bmatrix} w & y \\ x & z \end{bmatrix} \cdot \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} wa + yb & wc + yd \\ xa + zb & xc + zd \end{bmatrix}$$

Thus: $(A \cdot B)^t = B^t \cdot A^t$