Lecture 4: Gaussian Elimination and Homogeneous Equations

Reduced Row Echelon Form

An augmented matrix associated to a system of linear equations is said to be in **Reduced Row Echelon Form** (RREF) if the following properties hold:

1. The first non-zero entry from the left in each non-zero row is 1, and is called the **leading 1** for that row.
2. All entries in any column containing a leading 1 are zero, except the leading 1 itself.
3. Each leading 1 is to the right of the leading 1s in the rows above it.
4. All rows consisting entirely of zeros are at the bottom of the matrix.

The following matrices are in RREF:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Definition

Recalling that each column of the coefficient matrix corresponds to a variable in the system of equations, we call each variable associated to a leading 1 in the RREF a **leading variable.**

A variable (if any) that is **not** a leading variable is called a **nonleading variable** or a **free variable**.

Thus, if the original system of equations has n variables, then n equals the number of leading variables plus the number of free variables.

Extremely Important Point. When describing the solution set, each free variable is replaced by an independent parameter.

Thus, the number of independent parameters describing the solution set equals the number of free variables.

Equivalently: The number of parameters describing the solution set to a system of m equations in n unknowns equals n minus the number of leading 1s in the RREF of the augmented matrix.

Example

Solve the given system of equations by rendering the associated augmented matrix into RREF.

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\ 2x_1 - 4x_2 + x_3 &= 5 \\ x_1 - 2x_2 + 2x_3 - 3x_4 &= 4 \end{aligned}$$

Solution:

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right] \xrightarrow[-1\cdot R_1+R_3]{-2\cdot R_1+R_2} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}\cdot R_2} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right] \xrightarrow[1\cdot R_2+R_1]{-3\cdot R_2+R_3} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -0 & 0 \end{array}\right]$$

Note that x_1, x_3 are leading variables and x_2, x_4 are free variables.

Example continued

Thus, using independent parameters t and s, we have

$$x_1 = 2 + 2t - s,\ x_2 = t,\ x_3 = 1 + 2s,\ x_4 = s.$$

In terms of a solution set, we have

$$\{(2 + 2t - s, t, 1 + 2s, t) \mid s, t \in \mathbb{R}\}.$$

Algorithm for Gaussian elimination

The following steps lead effectively to the RREF of the augmented matrix:

1. Find the first column from the left containing a non-zero entry, say a, and interchange the row containing a with the first row.In this first step, a will more often than not be in the first row, first column of the augmented matrix.
2. Divide R_1 by a, so that the leading entry of R_1 is now 1.
3. Subtract multiples of R_1 from the rows below R_1 so that every entry in the matrix below the 1 in R_1 is 0.
4. Find the next column from the left containing a non-zero entry, say b, and interchange the row containing b with R_2. Now divide R_2 by b to get leading entry 1.
5. Use the leading 1 in R_2 to get 0s above and below it.
6. Continue in this fashion until arriving at the RREF.

If at any point in the process, we have a row consisting entirely of 0s, such a row should be moved to the bottom of the matrix.

Example

Solve the system by reducing the augmented matrix to RREF.

$$3x_1 + 7x_2 - x_3 = -1$$
$$x_1 + 3x_2 + x_3 = 1$$
$$-x_1 - 2x_2 + x_3 = 1.$$

Solution: $\left[\begin{array}{ccc|c} 3 & 7 & -1 & -1 \\ 1 & 3 & 1 & 1 \\ -1 & -2 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 3 & 7 & -1 & -1 \\ -1 & -2 & 1 & 1 \end{array}\right]$

$$\xrightarrow[-3 \cdot R_1 + R_2]{R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & -2 & -4 & -4 \\ 0 & 1 & 2 & 2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & -2 & -4 & -4 \end{array}\right]$$

$$\xrightarrow{2 \cdot R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-3 \cdot R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0. \end{array}\right].$$

Note, x_1, x_2 are leading variables and x_3 is a free variable. Thus, $x_1 = -5 + 5t,\ x_2 = 2 - 2t,\ x_3 = t$, for all $t \in \mathbb{R}$.

Class Example

Use Gaussian elimination to solve the system by putting the augmented matrix into RREF:

$$\begin{aligned} -2x + 3y + 3z &= -9 \\ 3x - 4y + z &= 5 \\ -5x + 7y + 2z &= -4 \end{aligned}$$

Solution: $\left[\begin{array}{ccc|c} -2 & 3 & 3 & -9 \\ 3 & -4 & 1 & 5 \\ -5 & 7 & 2 & -4 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{ccc|c} 1 & -1 & 4 & -4 \\ 3 & -4 & 1 & 5 \\ -5 & 7 & 2 & -4 \end{array}\right]$

$$\xrightarrow[5\cdot R_1+R_3]{-3\cdot R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 4 & -4 \\ 0 & -1 & -11 & 17 \\ 0 & 2 & 22 & -24 \end{array}\right] \xrightarrow{2\cdot R_2+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 4 & -4 \\ 0 & -1 & -11 & 17 \\ 0 & 0 & 0 & -10 \end{array}\right].$$

Thus, the system has no solution.

Definition

A system of linear equations is said to be **homogeneous** if the right hand side of each equation is zero, i.e., each equation in the system has the form

$$(*) \quad a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0.$$

Note that $x_1 = x_2 = \cdots = x_n = 0$ is always a solution to a homogeneous system of equations, called the **trivial solution.**

Any other solution is a **non-trivial solution.**

Two Important Properties. 1. **Sums of solutions are solutions.**

Suppose $(s_1, \ldots, s_n)$ and $(s'_1, \ldots, s'_n)$ are solutions to (*). Then

$$a_1s_1 + \cdots + a_ns_n = 0$$
$$a_1s'_1 + \cdots + a_ns'_n = 0$$

Adding, we get: $a_1(s_1 + s'_1) + \cdots + a_n(s_n + s'_n) = 0$, so that $(s_1 + s'_1, \ldots, s_n + s'_n)$ is also a solution.

2. **A scalar multiple of a solution to (*) is a solution.** Suppose $(s_1, \ldots, s_n)$ is a solution, so that

$$(**) \quad a_1 s_1 + \cdots + a_n s_n = 0.$$

Let $\lambda \in \mathbb{R}$. By a *scalar multiple* of a solution, we mean

$$\lambda \cdot (s_1, \ldots, s_n) = (\lambda \cdot s_1, \ldots, \lambda s_n).$$

If we multiply (**) above by λ we get

$$a_1 \cdot (\lambda s_1) + \cdots + a_n \cdot (\lambda s_n) = 0,$$

which shows that $(\lambda s_1, \ldots, \lambda s_n)$ is a solution to (*).

Important Consequence: Sums and scalar multiples of solutions to a homogenous system of linear equations are again solutions to the same system of equations.

Definition

Let $v_1 = (s_1^1, \ldots, s_n^1), v_2 = (s_1^2, \ldots, s_n^2), \ldots, v_k = (s_1^k, \ldots, s_n^k)$ be solutions to a homogeneous system of m equations in n unknowns. A *linear combination* of $v_1, \ldots, v_k$ is any expression of the form

$$\lambda_1 v_1 + \cdots + \lambda_k v_k,$$

with each $\lambda_i \in \mathbb{R}$.

If we write this out, we see that

$$\lambda_1 v_1 + \cdots \lambda_k v_k = (\lambda_1 s_1^1 + \cdots + \lambda_k s_1^k, \ldots, \lambda_1 s_n^1 + \cdots + \lambda_k s_n^k).$$

For example, if $v_1 = (1, 2, 3), v_2 = (0, 1, 1), v_3 = (1, -1, 6)$ are solutions to a homogeneous systems of equations in three variables, we calculate a linear combination as follows:

$$\begin{aligned} 2 \cdot v_1 + -4 \cdot v_2 + 8 \cdot v_3 &= 2 \cdot (1, 2, 3) + -4 \cdot (0, 1, 1) + 8 \cdot (1, -1, 6) \\ &= (2, 4, 6) + (0, -4, -4) + (8, -8, 48) \\ &= (10, -8, 50). \end{aligned}$$

Theorem

Suppose $v_1, \ldots, v_k \in \mathbb{R}^n$ are solutions to a **homogeneous** *system of m linear equations in n unknowns. Then, any linear combination $\lambda_1 v_1 + \cdots + \lambda_k v_k$ is also a solution.*

Moreover, given any homogenous system of m linear equations in n unknowns, there exist solutions, i.e., vectors $v_1, \ldots, v_k$, in $\mathbb{R}^n$ such that **every** *solution to the system is a linear combination of $v_1, \ldots, v_k$.*

Comments. 1. The first part of the theorem follows by combining the two Important Points from above.

2. By taking each $\lambda_j = 0$ above, one gets the zero solution, which is always a solution to any homogeneous system of linear equations.

3. It may be that the zero solution is the only solution, which is still consistent with the statement of the theorem.

Example

In this example we illustrate how to find the so-called **basic solutions** to a homogeneous system of linear equations. Suppose a given system led to the following RREF of the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right].$$

Thus, x_1 and x_3 are the leading variables and x_2, x_4 are the free variables. The solutions are:

$$x_1 = -2s + t, x_2 = s, x_3 = -4t, x_4 = t,$$

for all $s, t \in \mathbb{R}$.

Example continued

As a solution set, the solutions are $\{(-2s+t, s, -4t, t) \mid s, t \in \mathbb{R}\}$.

Alternately, the solutions are all exprssions of the form $\begin{bmatrix} -2s+t \\ s \\ -4t \\ t \end{bmatrix}$, for all $s, t \in \mathbb{R}$. . We can write this last expression as a linear combination:

$$(\star) \quad \begin{bmatrix} -2s+t \\ s \\ -4t \\ t \end{bmatrix} = s \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 1 \\ 0 \\ -4 \\ 1 \end{bmatrix},$$

where s, t are arbitrary real numbers.

Example continued

Thus, every solution to the original system of equations is a linear combination of the vectors

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 1 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

We call the vectors (solutions) v_1, v_2 **basic solutions** to the system of equations.

Note that the basic solutions are in fact solutions, since v_1 is obtained by taking $s = 1, t = 0$ in $(\star)$ and v_2 is obtained by taking $s = 0, t = 1$ in $(\star)$.

Class Example

Find two basic solutions to the homogeneous system of equations having augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 0 \\ 0 & 1 & 0 & 4 & 0 \end{array}\right].$$

Solution: Note that x_1, x_2 are the leading variables and x_3, x_4 are the free variables. Thus, in vector form the solutions are all vectors $\begin{bmatrix} -3s+t \\ -4t \\ s \\ t \end{bmatrix}$, with $s, t \in \mathbb{R}$ arbitrary. Separating this expression into two terms, we get

$$\begin{bmatrix} -3s+t \\ -4t \\ s \\ t \end{bmatrix} = s \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}.$$

Class Example continued

Thus, $v_1 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$ are basic vectors for the solutions to the original system of equations.

We will see later in the semester, that basic vectors for the solution space of a homogeneous system of linear equations need not be unique.

For example, there are infinitely many pairs of basic vectors for the solution space of the system above.

Class Example

Express the solution to the homogenous system with the following augmented matrix in terms of basic solutions:

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 2 & 1 & 0 \\ 1 & 2 & 2 & -1 & 1 & 0 \\ 3 & 6 & 0 & 3 & 3 & 0 \end{array}\right].$$

Solution: $\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 2 & 1 & 0 \\ 1 & 2 & 2 & -1 & 1 & 0 \\ 3 & 6 & 0 & 3 & 3 & 0 \end{array}\right] \xrightarrow[-3\cdot R_1+r_3]{-R_1+R_2} \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 2 & 1 & 0 \\ 0 & 0 & 3 & -3 & 0 & 0 \\ 0 & 0 & 3 & -3 & 0 & 0 \end{array}\right]$

$$\xrightarrow{\frac{1}{3}\cdot R_2} \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 3 & -3 & 0 & 0 \end{array}\right] \xrightarrow[R_2+R_1]{-3\cdot R_2+R_3} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, x_1, x_3 are the leading variables and x_2, x_4, x_5 are the free variables. The solutions are:

$$x_1 = -2s - t - u, x_2 = s, x_3 = t, x_4 = t, x_5 = u,$$

for all real numbers s, t, u.

Class Example continued

In terms of the basic solutions we have:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - t - u \\ s \\ t \\ t \\ u \end{bmatrix} = s \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + u \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$