Lecture 3: Gaussian Elimination, continued

Definition

The process of solving a system of linear equations by converting the system to an augmented matrix is called *Gaussian Elimination*.

The general strategy is as follows:

- Convert the system of linear equations into an augmented matrix.
- Perform various *Elementary Row Operations* on the augmented matrix until it is in a desirable form.
- Read off or write down the solution set.

In solving a system of linear equations using Gaussian Elimination, we are free to convert back to a system of linear equations from any augmented matrix that arises during the elimination process.

Crucial Fact: Any new system of equations that arises along the way has the same solution set as the original system of equations.

Definition

Elementary Row operations:

- Interchange two rows: $R_i \leftrightarrow R_j$
- Multiple a row by a **non-zero** number: R_j becomes $\lambda \cdot R_j$, $\lambda \neq 0$
- Add a multiple of one row to another row: R_i becomes $R_i + \lambda \cdot R_j$.

Crucial Fact Revisited: Given a system of linear equations, the solution set does not change as we perform elementary row operations on the corresponding augmented matrix.

General Principle

If at any stage in the process of Gaussian elimination, we arrive at an augmented matrix having a row of the form

$$[0\ 0\ 0\ \cdots\ 0 \mid \delta],$$

with $\delta \neq 0$, we may stop and conclude that the system has **no solution**.

Otherwise, the system will have a unique solution or infinitely many solutions.

Conversely, if the system has no solution, Gaussian elimination will always lead to a row of the form

$$[0\ 0\ 0\ \cdots\ 0 \mid \delta],$$

with $\delta \neq 0$.

General Principle continued

ALTERNATELY: The system **has** a solution if and only if Gaussian elimination does **NOT** lead to a row of the form

$$[0\ 0\ 0\ \cdots\ 0 \mid \delta],$$

with $\delta \neq 0$.

Such a system is said to be *CONSISTENT*.

Second General Principle

If we have a consistent system of linear equations in n variables, then the system will have a **unique solution** if and only if we can put the augmented matrix into the form

$$\left[\begin{array}{ccccc|c} 1 & * & * & \cdots & * & * \\ 0 & 1 & * & \cdots & * & * \\ 0 & 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & * \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \end{array}\right],$$

with n 1s down the main diagonal and zeros below.

Important: In this case, the number of *leading ones* equals the number of variables.

Third Scenario

If neither principle holds, the system has infinitely many solutions.

For example, if the system leads to the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 99.99 \\ 0 & 1 & 0 & 1 & 0 & \sqrt{7} \end{array}\right]$$

then there are infinitely many solution.

So, if the original system had variables x, y, z, w, u, then the augmented matrix yields $x = 99.99 - z - u$ and $y = \sqrt{7} - w$.

Introducing the independent parameters t_1, t_2, t_3 to replace z, w, u respectively, the solution set would be

$$\{((99.9 - t_1 - t_3, \sqrt{7} - t_2, t_1, t_2, t_3) \mid t_i, t_2, t_3 \in \mathbb{R}\}.$$

Class Example

Use Gaussian elimination to determine which system has infinitely many solutions and which system has no solution.

$$\begin{aligned} 2x + 3y &= 6 \\ 6x + 9y &= 18 \end{aligned} \qquad \begin{aligned} x + 4y &= 9 \\ 6x + 24y &= 40 \end{aligned}$$

Solution: For the first system we have

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 6 & 9 & 18 \end{array}\right] \xrightarrow{\frac{1}{2}\cdot R_1} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 3 \\ 6 & 9 & 18 \end{array}\right] \xrightarrow{-6\cdot R_1 + R_2} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 3 \\ 0 & 0 & 0 \end{array}\right],$$

and thus the first system has infinitely many solutions. For the second system

$$\left[\begin{array}{cc|c} 1 & 4 & 9 \\ 6 & 24 & 40 \end{array}\right] \xrightarrow{-6\cdot R_1 + R_2} \left[\begin{array}{cc|c} 1 & 4 & 9 \\ 0 & 0 & -14 \end{array}\right],$$

and thus the second system has no solution.

Class Example

Describe the nature of the solutions to the systems of equations associated to the following matrices:

$$A = \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right], B = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 \\ 4 & 5 & 6 & 7 \end{array}\right], C = \left[\begin{array}{cccc|c} 1 & -7 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: The system represented by A has a unique solution, $(6, 7)$.

The system represented by B has no solution.

The system represented by C has infinitely many solutions. The solution set is $\{(1 + 7t_1 - t_2, t_1, t_2, 3) \mid t_1, t_2 \in \mathbb{R}\}$.

Reduced Row Echelon Form

An augmented matrix associated to a system of linear equations is said to be in **Reduced Row Echelon Form** (RREF) if the following properties hold:

1. The first non-zero entry from the left in each non-zero row is 1, and is called the **leading 1** for that row.
2. All entries in any column containing a leading 1 are zero, except the leading 1 itself.
3. Each leading 1 is to the right of the leading 1s in the rows above it.
4. All rows consisting entirely of zeros are at the bottom of the matrix.

The following matrices are in RREF:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Goal of Gaussian Elimination

Put the augmented matrix associated to a system of linear equations into RREF and write down the solutions.

For the matrix $\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$, the solutions are:

$$x_1 = 2 - 2t_1 - 3t_2,\ x_2 = t_1,\ x_3 = 3 - 4t_2,\ x_4 = t_2,\ x_5 = 4,$$

for all real numbers t_1, t_2.

The second system has no solution.

The solutions to the system corresponding to $\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ are:

$$x_1 = t_1,\ x_2 = t_2,\ x_3 = 3,\ x_4 = 2.$$

Definition

Recalling that each column of the coefficient matrix corresponds to a variable in the system of equations, we call each variable associated to a leading 1 in the RREF a **leading variable.**

A variable (if any) that is **not** a leading variable is called a **nonleading variable** or a **free variable**.

Thus, if the original system of equations has n variables, then n equals the number of leading variables plus the number of free variables.

Extremely Important Point. When describing the solution set, each free variable is replaced by an independent parameter.

Thus, the number of independent parameters describing the solution set equals the number of free variables.

Equivalently: The number of parameters describing the solution set to a system of m equations in n unknowns equals n minus the number of leading 1s in the RREF of the augmented matrix.

Example

Solve the given system of equations by rendering the associated augmented matrix into RREF.

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\ 2x_1 - 4x_2 + x_3 &= 5 \\ x_1 - 2x_2 + 2x_3 - 3x_4 &= 4 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right] \xrightarrow[-1\cdot R_1+R_3]{-2\cdot R_1+R_2} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}\cdot R_2} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right] \xrightarrow[1\cdot R_2+R_1]{-3\cdot R_2+R_3} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -0 & 0 \end{array}\right]$$

Note that x_1, x_3 are leading variables and x_2, x_4 are free variables.

Example continued

Thus, using independent parameters t and s, we have

$$x_1 = 2 + 2t - s,\ x_2 = t,\ x_3 = 1 + 2s,\ x_4 = s.$$

In terms of a solution set, we have

$$\{(2 + 2t - s, t, 1 + 2s, t) \mid s, t \in \mathbb{R}\}.$$

Algorithm for Gaussian elimination

The following steps lead effectively to the RREF of the augmented matrix:

1. Find the first column from the left containing a non-zero entry, say a, and interchange the row containing a with the first row.In this first step, a will more often than not be in the first row, first column of the augmented matrix.
2. Divide R_1 by a, so that the leading entry of R_1 is now 1.
3. Subtract multiples of R_1 from the rows below R_1 so that every entry in the matrix below the 1 in R_1 is 0.
4. Excluding the first column, find the first column from the left containing a non-zero entry, say b, and interchange the row containing b with R_2. Now divide R_2 by b to get leading entry 1.
5. Use the leading 1 in R_2 to get 0s above and below it.
6. Continue in this fashion until arriving at the RREF.

Comments

1. If at any point in the process above, one arrives at a row consisting entirely of 0s, such a row should be moved to the bottom of the matrix.

2. If at any point one arrives at a row of the form $\begin{bmatrix} 0 & 0 & \cdots & 0 \mid d \end{bmatrix}$ with $d \neq 0$, STOP, since the system then has no solutions.

Class Example

Solve the system by reducing the augmented matrix to RREF.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -x_1 - 2x_2 + x_3 &= 1 \\ 3x_1 + 7x_2 - x_3 &= -1. \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ -1 & -2 & 1 & 1 \\ 3 & 7 & -1 & -1 \end{array}\right] \xrightarrow[-3\cdot R_1 + R_3]{R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & -2 & -4 & -4 \end{array}\right]$$

$$\xrightarrow{2\cdot R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-3\cdot R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0. \end{array}\right].$$

Note, x_1, x_2 are leading variables and x_3 is a free variable. Thus,

$$x_1 = -5 + 5t,\ x_2 = 2 - 2t,\ x_3 = t,$$

for all $t \in \mathbb{R}$.