Lecture 2: Gaussian Elimination

Example

Let us start with an example. Consider the system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 12 \\ 3y + 6z &= 18 \end{aligned}$$

If we divide the first equation by 2 and the second equation by 3, we get:

$$\begin{aligned} x + 2y + 3z &= 6 \\ y + 2z &= 6. \end{aligned}$$

Next, eliminate y by multiplying the second equation by -2 and adding it to the first equation:

$$\begin{aligned} x + (-z) &= -6 \\ y + 2z &= 6. \end{aligned}$$

Thus: $x = -6 + z$ and $y = 6 - 2z$, for z any real number. The solution set is:

$$\{(-6 + t, 6 - 2t, t) \mid t \in \mathbb{R}\}.$$

Example Continued

Important observation: Until the last step, the variables x, y, z were functioning as place holders. We solved the system by working only with the coefficients.

So let us solve the system by just working with the coefficients, by putting them in a box (or array), called the *augmented matrix* of the system of equations

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 12 \\ 0 & 3 & 6 & 18 \end{array}\right].$$

The large left portion $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 6 \end{bmatrix}$ of the augmented matrix is called the *coefficient matrix*.The right portion $\begin{bmatrix} 12 \\ 18 \end{bmatrix}$ of the augmented matrix is called the *constant matrix*.

Example Continued

Further things to notice about the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 12 \\ 0 & 3 & 6 & 18 \end{array}\right].$$

- Each column in the left portion of the matrix corresponds to a variable in the given system.
- Each row in the augmented matrix corresponds to an equation in the given system.
- We put 0 in a row if a variable does not appear in the expected place in the corresponding equation.

Example Continued

Now let's perform the same steps on the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 12 \\ 0 & 3 & 6 & 18 \end{array}\right]$$

that we did to solve the system of equations.

Step 1: Divide the first row by 2 and the second row by 3.

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 12 \\ 0 & 3 & 6 & 18 \end{array}\right] \xrightarrow[\frac{1}{3}\cdot R_2]{\frac{1}{2}\cdot R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 6 \end{array}\right].$$

Step 2: Multiply the second row by -2 and add it to the first row.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 6 \end{array}\right] \xrightarrow{-2\cdot R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -6 \\ 0 & 1 & 2 & 6 \end{array}\right].$$

Note, in the second step, we did not change R_2, we just used it to change R_1.

Example Continued

Convert back to a system of equations:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -6 \\ 0 & 1 & 2 & 6 \end{array}\right] \rightarrow \begin{array}{r} x + \ 0y + -z = \ \ \ 6 \\ 0x + -y + \ 2z = -6 \end{array}$$

which is the system

$$\begin{aligned} x + (-z) &= -6 \\ y + 2z &= 6. \end{aligned}$$

And thus, as before, $x = -6 + z$ and $y = 6 - 2z$, for z any real number.

The solution set is: $\{(-6 + t, 6 - 2t, t) \mid t \in \mathbb{R}\}$.

Important Point

Any system of m linear equations in n unknowns gives rise to an augmented matrix with m rows and $n + 1$ columns.

And conversely, any augmented matrix with m rows and $n + 1$ columns gives rise to a system of m linear equations in n unknowns.

Thus the systems of equations

$$\begin{aligned} 3x_1 - 5x_3 + 2x_4 &= 9 \\ 2x_2 + 9x_3 - 4x_4 &= -6 \\ x_2 + x_3 + x_4 &= 9 \end{aligned}$$

gives rise to the augmented matrix

$$\left[\begin{array}{cccc|c} 3 & 0 & -5 & 4 & 9 \\ 0 & 2 & 9 & -4 & -6 \\ 0 & 1 & 1 & 1 & 9 \end{array}\right].$$

Important Point Continued

And similarly, the augmented matrix

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 1 & 0 & \sqrt{7} \\ 1 & 0 & 1 & 0 & 1 & 99.99 \end{array}\right]$$

Gives rise to the system of equations

$$\begin{aligned} x_2 + x_4 &= \sqrt{7} \\ x_1 + x_3 + x_5 &= 99.99. \end{aligned}$$

Class Example

Convert

$$\begin{aligned} 2x - 3y + 5z &= 7 \\ 3x - 2z &= 9 \\ y + 8z &= 4 \end{aligned}$$

to an augmented matrix and convert $\left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & -2 & -2 & 6 \end{array}\right]$ to a system of equations.

Solution: The augmented matrix is $\left[\begin{array}{ccc|c} 2 & -3 & 5 & 7 \\ 3 & 0 & -2 & 9 \\ 0 & 1 & 8 & 4 \end{array}\right]$, and the system of equations is

$$\begin{aligned} 2x &= 4 \\ -2y - 2z &= 6. \end{aligned}$$

Definition

The process of solving a system of linear equations by converting the system to an augmented matrix is called *Gaussian Elimination*.

The general strategy is as follows:

- Convert the system of linear equations into an augmented matrix.
- Perform various *allowable* operations on the augmented matrix until it is in a desirable form.
- Read off or write down the solution set.

In solving a system of linear equations using Gaussian Elimination, we are free to convert back to a system of linear equations from any augmented matrix that arises during the elimination process.

Crucial Fact: Any new system of equations that arises along the way has the same solution set as the original system of equations.

Example

Let's look at the case of two equations in two unknowns

$$ax + by = c$$
$$dx + ey = f$$

The simplest version of this is

$$x = u$$
$$y = v,$$

which is easy to solve! The corresponding augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 0 & u \\ 0 & 1 & v \end{array}\right].$$

This suggests we should try to put the initial augmented matrix into this form.

Example

Use Gaussian Elimination to solve the system

$$\begin{aligned} 2x + 3y &= 1 \\ 3x + 2y &= -1. \end{aligned}$$

The augmented matrix is $\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 3 & 2 & -1 \end{array}\right]$. We would like to have a 1 in the upper left corner. Dividing R_1 by 2 is messy! Instead: Switch rows and then subtract

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 3 & 2 & -1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 3 & 2 & -1 \\ 2 & 3 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cc|c} 1 & -1 & -2 \\ 2 & 3 & 1 \end{array}\right].$$

Next: to get 0 in the lower left corner, add a multiple of R_1 to R_2.

$$\left[\begin{array}{cc|c} 1 & -1 & -2 \\ 2 & 3 & 1 \end{array}\right] \xrightarrow{-2 \cdot R_1 + R_2} \left[\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 5 & 5 \end{array}\right]$$

Example Continued

We now want 1 in the lower right corner of the coefficient matrix

$$\left[\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 5 & 5 \end{array}\right] \xrightarrow{\frac{1}{5}\cdot R_2} \left[\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 1 & 1 \end{array}\right].$$

Finally, add R_2 to R_1 to get:

$$\left[\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{2\cdot R_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right].$$

This last augmented matrix corresponds to the system

$$\begin{aligned} x &= -1 \\ y &= 1, \end{aligned}$$

so the solution set is $\{(-1, 1)\}$. Check:

$$\begin{aligned} 2(-1) + 3 \cdot 1 &= 1 \\ 3(-1) + 2 \cdot 1 &= -1. \end{aligned}$$

In general, what are the allowable operations on the augmented matrix associated to a system of linear equations?

Definition

Elementary Row operations:

- Interchange two rows: $R_i \leftrightarrow R_j$
- Multiple a row by a **non-zero** number: R_j becomes $\lambda \cdot R_j$, $\lambda \neq 0$
- Add a multiple of one row to another row: R_i becomes $R_i + \lambda \cdot R_j$.

Crucial Fact Revisited: Given a system of linear equations, the solution set does not change as we perform elementary row operations on the corresponding augmented matrix.

Example

Use Gaussian Elimination to solve the system

$$3x - 2y = 14$$
$$4x + 3y = 13$$

$$\left[\begin{array}{cc|c} 3 & -2 & 14 \\ 4 & 3 & 13 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 4 & 3 & 13 \\ 3 & -2 & 14 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cc|c} 1 & 5 & -1 \\ 3 & -2 & 14 \end{array}\right]$$

$$\xrightarrow{-3 \cdot R_1 + R_2} \left[\begin{array}{cc|c} 1 & 5 & -1 \\ 0 & -17 & 17 \end{array}\right] \xrightarrow{-\frac{1}{17} \cdot R_2} \left[\begin{array}{cc|c} 1 & 5 & -1 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{-5 \cdot R_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \end{array}\right]$$

Thus, $x = 4$ and $y = -1$. Check:

$$3 \cdot 4 - 2 \cdot (-1) = 14$$
$$4 \cdot 4 + 3 \cdot (-1) = 13$$

Class Example

Use Gaussian Elimination to solve the system

$$2x + 4y = 16$$
$$5x + 2y = 16.$$

Solution: Converting to an augmented matrix we have:

$$\left[\begin{array}{cc|c} 2 & 4 & 16 \\ 5 & 2 & 16 \end{array}\right] \xrightarrow{\frac{1}{2}\cdot R_1} \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 5 & 2 & 16 \end{array}\right] \xrightarrow{-5\cdot R_1 + R_2} \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -8 & -24 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -8 & -24 \end{array}\right] \xrightarrow{-\frac{1}{8}\cdot R_2} \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{-2\cdot R_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

Thus, $x = 2$ and $y = 3$, or as a set, $\{(2, 3)\}$. Check:

$$2 \cdot 2 + 4 \cdot 3 = 16$$
$$5 \cdot 2 + 2 \cdot 3 = 16.$$

Example

The previous examples illustrate how the augmented matrix indicates that a system has a unique solution. There are also the cases of no solution or infinitely many solutions.

Case 1: The augmented matrix has the form $\left[\begin{array}{cc|c} \alpha & \beta & \gamma \\ 0 & 0 & \delta \end{array}\right]$, with $\delta \neq 0$. In this case, the second row yields the equation, $0y = \delta$, i.e., $0 = \delta$, which is a contradiction.. **Thus the system has no solution.**

Case 2: The augmented matrix has the form $\left[\begin{array}{cc|c} \alpha & \beta & \gamma \\ 0 & 0 & 0 \end{array}\right]$, with α or β non-zero. In this case, the system has been reduced to just one equation: $\alpha x + \beta y = \gamma$, which has **infinitely many solutions**.

Then, as we have seen previously, if say, $\alpha \neq 0$, the solution set is $\{(\frac{\gamma}{\alpha} - \frac{\beta}{\alpha}t, t) \mid t \in \mathbb{R}\}$.

If $\alpha = 0$ and $\beta \neq 0$, then the solution set is $\{(t, \frac{\gamma}{\beta}) \mid t \in \mathbb{R}\}$.

Class Example

Use Gaussian elimination to determine which system has infinitely many solutions and which system has no solution.

$$\begin{aligned} 2x + 3y &= 6 \\ 6x + 9y &= 18 \end{aligned} \qquad \begin{aligned} x + 4y &= 9 \\ 6x + 24y &= 40 \end{aligned}$$

Solution: For the first system we have

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 6 & 9 & 18 \end{array}\right] \xrightarrow{-3\cdot R_1 + R_2} \left[\begin{array}{cc|c} 2 & 3 & 6 \\ 0 & 0 & 0 \end{array}\right],$$

and thus the first sytem has infinitely many solutions. For the second system

$$\left[\begin{array}{cc|c} 1 & 4 & 9 \\ 6 & 24 & 40 \end{array}\right] \xrightarrow{-6\cdot R_1 + R_2} \left[\begin{array}{cc|c} 1 & 4 & 9 \\ 0 & 0 & -14 \end{array}\right],$$

and thus the second system has no solution.

General Principle

If at any stage in the process of Gaussian elimination, we arrive at an augmented matrix having a row of the form

$$[0\ 0\ 0\ \cdots\ 0 \mid \delta],$$

with $\delta \neq 0$, we may stop and conclude that the system has **no solution**.

Otherwise, the system will have a unique solution or infinitely many solutions.

Second General Principle

If we start with a system of linear equations in n variables that has a solution, then the system will have a **unique solution** if we can put the augmented matrix into the form

$$\left[\begin{array}{ccccc|c} 1 & * & * & \cdots & * & * \\ 0 & 1 & * & \cdots & * & * \\ 0 & 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & * \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \end{array}\right],$$

with n 1s down the main diagonal and zeros below.

Example

If neither principle holds, the system has infinitely many solutions, e.g.,

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 99.99 \\ 0 & 1 & 0 & 1 & 0 & \sqrt{7} \end{array}\right]$$

So, for example, if the original system had variables x, y, z, w, u, then the augmented matrix yields $x = 99.99 - z - u$ and $y = \sqrt{7} - w$.

Introducing the independent parameters t_1, t_2, t_3 to replace z, w, u respectively, the solution set would be

$$\{((99.9 - t_1 - t_3, \sqrt{7} - t_2, t_1, t_2, t_3) \mid t_i, t_2, t_3 \in \mathbb{R}\}.$$