Lecture 18: Bases and Dimension

Theorem

Very Important Theorem. Let $v_1, \ldots, v_r$ be vectors in $\mathbb{R}^n$ and let A denote the $n \times r$ matrix $A = [v_1 \ v_2 \ \cdots \ v_r]$. The following equivalent conditions are equivalent::

(i) No vector in the list can be written as a linear combination of the remaining vectors in the list.

(ii) If we have a linear combination

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_r v_r = \mathbf{0},$$

then $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$.

(iii) Any vector in span$\{v_1, \ldots, v_r\}$ can be written **uniquely** as a linear combination of $v_1, \ldots, v_r$.

(iv) The system of equations $A \cdot \mathrm{X} = \mathbf{0}$ has only the 0 solution.

Vectors satisfying the conditions above are said to be linearly independent.

Definition

Vectors $v_1, \ldots, v_r \in \mathbb{R}^n$ are linearly dependent if they are not linearly independent.

In particular: Vectors $v_1, \ldots, v_r$ are linearly dependent if one of the following equivalent conditions hold:

(i) Some v_i is in the span of $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r$.

(ii) There exists a non-trivial dependence relation:

$$\lambda_1 v_1 + \cdots + \lambda_r v_r = \mathbf{0}$$

with NOT all $\lambda_i = 0$.

(iii) For $A = [v_1 \ \cdots \ v_r]$, there is a non-zero solution to $A \cdot \mathrm{X} = \mathbf{0}$.

Observation: Suppose $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = \mathbf{0}$, with say, $\lambda_2 \neq 0$. Then

$$-\lambda_2 v_2 = \lambda_1 v_1 + \lambda_3 v_3,$$

so $v_2 = -\frac{\lambda_1}{\lambda_2} v_1 + -\frac{\lambda_3}{\lambda_1} v_3$.This shows how a dependence relation among the vectors v_i leads to expressing one of the vectors in terms of the others.

Example

The vectors $v_1 = \begin{bmatrix}1\\1\\2\end{bmatrix}, v_2 = \begin{bmatrix}1\\2\\1\end{bmatrix}, v_3 = \begin{bmatrix}3\\4\\5\end{bmatrix}$ are linearly **dependent**.

Find a dependence relation among them and use it to express one of the vectors as a linear combination of the remaining vectors.

Solution:If the vectors are not linearly independent, then there is a non-zero solution to the system $AX = \mathbf{0}$, where $A = [v_1\ v_2\ v_3]$.

$$\left[\begin{array}{ccc|c}1 & 1 & 3 & 0\\1 & 2 & 4 & 0\\2 & 1 & 5 & 0\end{array}\right] \xrightarrow[-2\cdot R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c}1 & 1 & 3 & 0\\0 & 1 & 1 & 0\\0 & -1 & -1 & 0\end{array}\right] \xrightarrow[R_2+R_3]{-R_2+R_1} \left[\begin{array}{ccc|c}1 & 0 & 2 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right].$$

We can write the solution to the homogeneous system as

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-2t\\-t\\t\end{bmatrix} = t\cdot\begin{bmatrix}-2\\-1\\1\end{bmatrix}.$$

Example continued

This shows that $-2 \cdot v_1 + (-1) \cdot v_2 + v_3 = \mathbf{0}$.Thus, $v_3 = 2 \cdot v_1 + v_2$.

CHECK: $2 \cdot v_1 + v_2 = 2 \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$, as required.

Since any multiple of $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ is also a solution, any such multiple gives a dependence relation on v_1, v_2, v_3. For example, taking $t = -3$, we get that $\begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix}$ is a solution, so that $6 \cdot v_1 + 3 \cdot v_2 - 3 \cdot v_3 = \mathbf{0}$.

Thus there are infinitely many dependence relations among v_1, v_2, v_3.

But in this case, just one way to write v_3 as a linear combination of v_1 and v_2.

Definition

Important Definition: Let $U \subseteq \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$. Vectors $v_1, \ldots, v_r \in U$ are a basis for U if:

(i) $U = \text{span}\{v_1, \ldots, v_r\}$.

(ii) The vectors $v_1, \ldots, v_r$ are linearly independent.

In particular: A basis for $\mathbb{R}^n$ is a collection of linearly independent vectors that span $\mathbb{R}^n$.

Moreover: If $v_1, \ldots, v_n$ is a basis for $\mathbb{R}^n$, then: Every vector in $\mathbb{R}^n$ can be written *uniquely* as a linear combination of $v_1, \ldots, v_n$.

Examples: (i) The standard basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is a basis for $\mathbb{R}^n$.

(ii) The basic solutions to a homogeneous system of linear equations form a basis for the solution space of that system.

(iii) If λ is an eigenvalue for the matrix A, then the basic λ-eigenvectors form a basis for E_λ, the eigenspace of λ.

Example

Find a basis for the subspace of $\mathbb{R}^3$ that is the solution space to the homogeneous equation:

$$2x - 4y + 10z = 0.$$

Note that this solution space is a plane through the origin in $\mathbb{R}^3$.

Solution: $x = 2y - 5z$. Thus, in vector form, the solutions are given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2s - 5t \\ s \\ t \end{bmatrix} = s \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}.$$

Thus $v_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$ are basic solutions.

Example continued

To see that the basic solutions are in fact a basis for the solution space, note that the vector equation shows that the basic solutions span the solution space.

In addition, if $\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 = \mathbf{0}$, then:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \alpha \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \cdot \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\alpha_1 - 5\alpha_2 \\ \alpha_1 \\ \alpha_2 \end{bmatrix},$$

which gives: $\alpha_1 = \alpha_2 = 0$.

Thus v_1, v_2 are linearly independent and therefore form a basis for the solutions space, or equivalently, a basis for the given plane through the origin.

Class Example

The matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ has 2 as an eigenvalue of multiplicity two.

Find a basis for the eigenspace E_2.

Solution: We find the solutions to the homogeneous system having $2I_3 - A = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ as its coefficient matrix.

This matrix clearly reduces to $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. For such a homogeneous system, $y = 0$, while x and z are free variables.Thus the solutions are:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Class Example continued

Thus, the basic solutions to the homogeneous system with coefficient matrix $2I_3 - A$ are: $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$, which in turn form a basis for the eigenspace E_2.

Comment

Very Important Fact. Suppose the subspace U of $\mathbb{R}^n$ is spanned by the vectors $v_1, \ldots, v_r$. Then there exists a subset of $v_1, \ldots, v_r$ forming a basis of U.

Why: Suppose $U = \text{span}\{v_1, v_2, v_3, v_4\}$. If $v_1, \ldots, v_4$ are linearly independent, they form a basis for U.

Otherwise of one the vectors is in the span of the remaining ones: say, $v_2 = av_1 + bv_3 + cv_4$.

Suppose $u \in U$. We can write

$$u = pv_1 + qv_2 + rv_3 + sv_4 = pv_1 + q(av_1 + bv_3 + cv_4) + rv_3 + sv_4$$

$$= (p + aq)v_1 + (qb + r)v_3 + (qc + s)v_4.$$

Thus, $u \in \text{span}\{v_1, v_3, v_4\}$.Thus: $U = \text{span}\{v_1, v_3, v_4\}$.

If v_1, v_3, v_4 are linearly independent, they form a basis for U. Otherwise, we may eliminate another vector and continue the process until we have a linearly independent spanning set for U, that is, a basis for U.

Example

Find a basis for the subspace $U = \text{span}\{v_1, v_2, v_3\}$, where
$v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix}$.

Solution: We consider $A \cdot \mathrm{X} = \mathbf{0}$, for $A = [v_1 \ v_2 \ v_3]$.

$$\left[\begin{array}{ccc|c} 2 & 2 & 8 & 0 \\ 1 & 2 & 6 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right] \xrightarrow{\frac{1}{2} \cdot R_1} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 1 & 2 & 6 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right] \xrightarrow[-1R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This shows that the vectors, v_1, v_2, v_3 are linearly dependent. The dependence relations are given by the solutions to $A \cdot \mathrm{X} = \mathbf{0}$, whic are:
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ -2t \\ t \end{bmatrix}$.

Example continued

Taking $t = 1$, obtain $-2v_1 - 2v_2 + v_3 = \mathbf{0}$, so $v_3 = 2v_1 + 2v_2$. This shows v_3 is redundant, and $U = \text{span}\{v_1, v_2\}$.

If v_1, v_2 are linearly independent, they will form a basis for U. We look at the solutions to $B \cdot \mathrm{X} = \mathbf{0}$, with $B = [v_1 \; v_2]$.

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow[-R_2+R_3]{-R_2+R_1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & -1 & 0 \end{array}\right] \xrightarrow[-R_3]{-R_1+R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow[-R_2+R_3]{\frac{1}{2}\cdot R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus, the system $B \cdot \mathrm{X} = \mathbf{0}$ has only the $\mathbf{0}$ solution, so that v_1, v_2 are linearly independent.
Alternately: v_1, v_2 are linearly independent if they are are NOT linearly dependent.So: suppose v_1 is in the span of v_2, i.e, $v_1 = \lambda \cdot v_2$.

Then $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \lambda \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. From the first entry we get $2 = \lambda \cdot 2$, so $\lambda = 1$.
But comparing second entries, $1 = 1 \cdot 2$ a contradiction. Thus, v_1, v_2 are independent, and v_1, v_2 form a basis for U.

Example

Find a basis for the subspace $V = \text{span}\{v_1, v_2, v_3\}$, for
$v_1 = \begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} -4 \\ -2 \\ -6 \end{bmatrix}$.

Solution: By inspection, we see that $v_1 = 4v_2$, so that v_1 is redundant, and $V = \text{span}\{v_2, v_3\}$.

Now note that $v_3 = -2 \cdot v_2$.Thus, v_3 is redundant, and $V = \text{span}\{v_2\}$.

Thus, v_2 is a basis for V.

MAIN POINT REITERATED. Given a spanning set for a subspace, we may throw out redundant spanning vectors until we have a linearly independent spanning set – which is then a basis for that subspace.

Class Example

Find a basis for the subspace U of $\mathbb{R}^4$ spanned by the vectors
$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ -5 \\ 2 \\ 5 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: We Start with the equation $A \cdot \mathrm{X} = \mathbf{0}$, with $A = [v_1 \ v_2 \ v_3]$.

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ -1 & -5 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 5 & 1 & 0 \end{array}\right] \xrightarrow[-R_1+R_4]{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \xrightarrow[-R_2+R_4]{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2} \cdot R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$ The solutions to $A \cdot \mathrm{X} = \mathbf{0}$ are $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3t \\ -\frac{t}{2} \\ t \\ 0 \end{bmatrix}$.

Class Example continued

Taking $t = 1$, we have the dependence relation

$$-3v_1 - \frac{1}{2}v_2 + v_3 = \mathbf{0}. \qquad (*)$$

Thus, $v_3 = 3v_1 + \frac{3}{2}v_2$. Therefore v_3 is redundant, so v_1, v_2 span U.
To see v_1, v_2 are linearly independent, suppose $v_1 = \lambda v_2$.

Then $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \lambda \cdot \begin{bmatrix} 3 \\ -2 \\ 2 \\ 5 \end{bmatrix}$. From the 3rd coordinate: $\lambda = 0$.

The first coordinate becomes $1 = 0 \cdot 3$, a contradiction. Thus, v_1, v_2 are not DEPENDENT, so they are independent. Therefore, v_1, v_2 form a basis for U.

NOTE: The same argument using (*) shows that v_1, v_3 and v_2, v_3 are also bases for U.

Fundamental Theorem

Let $v_1, \ldots, v_r$ be vectors in $\mathbb{R}^n$ that span the subspace U and suppose $w_1, \ldots, w_t \in U$ are linearly independent. Then:

(i) $t \leq r$**.** In other words:

   In any given subspace, the number of linearly independent vectors is always less than or equal to the number of spanning vectors.

(ii) **Any two bases for U have the same number of elements.**
   Why: If $v_1, \ldots, v_r$ and $w_1, \ldots, w_t$ are bases for U, then $t \leq r$ since the w's are linearly independent and the v's span U.

   On the other hand, $r \leq t$, since the v's are linearly independent and the w's span U.

   Thus, $r = t$, and the two sets of bases have the same number of elements.

Definition

Let U be a subspace of $\mathbb{R}^n$. The **dimension** of U is the number of elements in any basis of U.

Corollary. The dimension of $\mathbb{R}^n$ equals n.

WHY: $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ forms a basis for $\mathbb{R}^n$.

Corollary. The dimensions of the subspaces of $\mathbb{R}^3$ are given as follows:

(i) $\{\mathbf{0}\}$ is zero dimensional – it does not have a basis.

(ii) A line L through the origin is one dimensional. Any vector on the line forms a basis for L.

(iii) A plane P through the origin is two dimensional. Any two non-collinear vectors in P form a basis for P.

Comment

Most examples of finding a basis for for a subspace of Euclidean space require one to delete redundant vectors from a spanning set.For other examples, let A be an $m \times n$ matrix:

(i) To find a basis for the image of A, start with the space spanned by the columns of A.

(ii) To find a basis for the null space of A, find a set of basic solutions to the homogeneous system $A \cdot \mathrm{X} = 0$.

(iii) If $m = n$, to find a basis for the eigenspace E_λ, find a set of basic solutions to the homogeneous system $(\lambda \cdot I_n - A) \cdot \mathrm{X} = 0$.