Lecture 16: Subspaces, Spanning, and Linear Independence

Definition

Recall that a subset $W \subseteq \mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if it satisfies the following properties:

(i) $\mathbf{0} \in W$.

(ii) If $v_1, v_2 \in W$, then $v_1 + v_2 \in W$.

(iii) if $v \in W$, then $\lambda v \in W$, for all $\lambda \in \mathbb{R}$.

In particular: If $v_1, \ldots, v_r$ are in W then any linear combination

$$\gamma_1 v_1 + \cdots + \gamma_r v_r$$

is also in W.

Example

Examples of subspaces of Euclidean spaces include:

(i) Solutions to any homogeneous system of linear equations.

(ii) The nullspace of an $m \times n$ matrix A, i.e., all vectors $v \in \mathbb{R}^n$ such that

$$Av = \mathbf{0}.$$

(iii) The image of A,i.e., all vectors in $\mathbb{R}^m$ of the form Av, as v ranges over $\mathbb{R}^n$.

(iv) If λ is an eigenvalue of A, the set of all λ-eigenvectors together with $\mathbf{0}$ is a subspace of $\mathbb{R}^n$, called the Eigenspace of λ.

(v) Given vectors $v_1, \ldots v_r$ in $\mathbb{R}^n$, the span of the vectors $v_1, \ldots, v_r$ is a subspace of $\mathbb{R}^r$.

Recall $\text{span}\{v_1, \ldots, v_r\}$ is the set of all linear combinations of the vectors $v_1, \ldots, v_r$.

Example

What are the subspaces of $\mathbb{R}^2$?

(i) $\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^2$.

(ii) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

(iii) Any line through the origin is a subspace of $\mathbb{R}^2$.

These are the only subspaces of $\mathbb{R}^2$.

Let L be a line in $\mathbb{R}^2$ through the origin. L is given by $y = mx$, where m is the slope of L.

Thus, L is the solution set of the homogeneous equation $x + (-m)y = 0$, which shows that L is a subspace.

Thus, if v is any vector on the line L, then $L = \text{span}\{v\}$.

Moreover, $\mathbb{R}^2$, when considered as a subspace equals $\text{span}\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$,

since $\begin{bmatrix}a\\b\end{bmatrix} = a \cdot \begin{bmatrix}1\\0\end{bmatrix} + b \cdot \begin{bmatrix}0\\1\end{bmatrix}$, for all $\begin{bmatrix}a\\b\end{bmatrix} \in \mathbb{R}^2$.

Example

The subspaces of $\mathbb{R}^3$ are:

(i) $\{\mathbf{0}\}$.

(ii) $\mathbb{R}^3$.

(iii) Any line through the origin.

(iv) Any plane through the origin.

Why: Each of these sets are the solution set to a homogeneous system of equations. For example:

(a) Any plane through the origin is defined by an equation of the form: $ax + by + cz = 0$, for fixed constants $a, b, c \in \mathbb{R}$.

(b) Any line through the origin is the intersection of two planes through the origin and is thus the solution set to a system of equations of the form:

$$\begin{aligned} ax + by + cz &= 0 \\ dx + ey + fz &= 0. \end{aligned}$$

Example continued

Note if $W \subseteq \mathbb{R}^3$ is a plane through the origin, then $W = \text{span}\{v_1, v_2\}$, for any two non-collinear vectors $v_1, v_2 \in W$.

Example

A line or plane in $\mathbb{R}^2$ or $\mathbb{R}^3$ that does not pass through the origin, is NOT a subspace. We require subspaces to contain $\mathbf{0}$.

Another reason: Suppose $L : y = mx + b$ is a line in $\mathbb{R}^2$, with $b \neq 0$. Let $v_1 = \begin{bmatrix} a \\ ma + b \end{bmatrix}$ and $v_2 = \begin{bmatrix} c \\ mc + b \end{bmatrix}$ be vectors in L. Then $v_1 + v_2 = \begin{bmatrix} a + c \\ m(a + c) + 2b \end{bmatrix}$, which does not belong to L.

Definition

Let $U \subseteq \mathbb{R}^n$ be a subspace. Vectors $v_1, \ldots, v_r \in U$ span U if U is the subspace spanned by $v_1, \ldots, v_r$,

In other words, U is the set of all linear combinations of $v_1, \ldots, v_r$.

In this case we write $U = \text{span}\{v_1, \ldots, v_r\}$.

The subspace U can have infinitely many different spanning sets.

Important Observation: Let $A = [v_1 \ v_2 \ \cdots \ v_r]$ be the $n \times r$ matrix whose columns are $v_1, \ldots, v_r$. Then:

$$u \in U \quad \text{if and only if} \quad u = A \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{bmatrix},$$

since

$$A \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{bmatrix} = [v_1 \cdots v_r] \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{bmatrix} = \lambda_1 v_1 + \cdots + \lambda_r v_r.$$

Example

To see this more explicitly: Let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Then a typical linear combination of v_1, v_2, v_3 has the form:

$$\lambda_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 + \lambda_2 \\ \lambda_1 + \lambda_3 \\ \lambda_2 + \lambda_3 \end{bmatrix}.$$

On the other hand, if $A = [v_1 \; v_2 \; v_3]$ is the matrix with columns v_1, v_2, v_3, then:

$$A \cdot \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 + \lambda_2 \\ \lambda_1 + \lambda_3 \\ \lambda_2 + \lambda_3 \end{bmatrix}.$$

To Reiterate: A matrix product Av is the vector obtained by taking the linear combination of the columns of A using the entries of v as coefficients.

Comment

Thus: A vector $b \in \mathbb{R}^n$ is in the span of $v_1, \ldots, v_r$ if and only if the $n \times r$ system of equations given by

$$A \cdot \mathrm{X} = b$$

has a solution.

Moreover, if $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{bmatrix}$ is a solution, then

$$b = \lambda_1 v_1 + \cdots + \lambda_r v_r.$$

Important Consequence: Given any $m \times n$ matrix A, the subspace that is the image of A, denoted $\mathrm{im}(A)$, is the subspace spanned of $\mathbb{R}^m$ spanned by the columns of A.

Example

Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Is $b = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ in $\text{span}\{v_1, v_2, v_3\}$?

Solution: Let $A = [v_1 \ v_2 \ v_3]$. We seek a solution to $A \cdot X = b$. We use Gaussian elimination.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-1 \cdot R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{array}\right] \xrightarrow{-1 \cdot R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2} \cdot R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \xrightarrow[-1 \cdot R_3 + R_2]{-1 \cdot R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right].$$

This shows that $b = \frac{3}{2} \cdot v_1 - \frac{1}{2} \cdot v_2 + \frac{1}{2} \cdot v_3.$,

Thus, $b \in \text{span}\{v_1, v_2, v_3\}$.

Example continued

CHECK: $\frac{3}{2} \cdot v_1 - \frac{1}{2} \cdot v_2 + \frac{1}{2} \cdot v_3 =$

$$\frac{3}{2} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ \frac{3}{2} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = b.$$

Example

Determine if the vector $b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ belongs to the subspace spanned by the vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

We seek a solution to the system given by $A \cdot \mathrm{X} = b$, where $A = [v_1 \ v_2 \ v_3]$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right],$$

which shows that the system has no solution. Therefore b is not in $\text{span}\{v_1, v_2, v_3\}$.

Class Example

Determine if $b = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$ belongs to span$\{v_1, v_2, v_3\}$, for

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

If so, write b as a linear combination of v_1, v_2, v_3.

Then, verify your answer.

Class Example continued

Solution: Using Gaussian Elimination,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 1 & 1 & 3 \\ 1 & 1 & 2 & 1 \end{array}\right] \xrightarrow[-1\cdot R_1+R_3]{-2\cdot R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -3 & -1 & -5 \\ 0 & -1 & 1 & -3 \end{array}\right] \xrightarrow{R_2\leftrightarrow -R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & -3 & -1 & -5 \end{array}\right]$$

$$\xrightarrow[3\cdot R_2+R_3]{-2\cdot R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -4 & 4 \end{array}\right] \xrightarrow{-\frac{1}{4}\cdot R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[1\cdot R_3+R_2]{-3\cdot R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Thus: $b = 1 \cdot v_1 + 2 \cdot v_2 - v_3$.

Class Example continued

CHECK: $1 \cdot v_1 + 2 \cdot v_2 - v_3 =$

$$1 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

Definition

Set $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ in $\mathbb{R}^n$.

The vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are called the standard basis for $\mathbb{R}^n$.

Note that $\text{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_n\} = \mathbb{R}^n$, since

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1 \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_n \cdot \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = a_1\mathbf{e}_1 + a_2\mathbf{e}_3 + \cdots + a_n\mathbf{e}_n,$$

for all $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

Properties of the standard basis

(i) No vector in the standard basis can be written as a linear combination of the remaining standard basis vectors.

Equivalently:

$$\mathbf{e}_i \notin \text{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \ldots, \mathbf{e}_n\},$$

for all $\leq i \leq n$.

For example, in $\mathbb{R}^3$, if we try to write $\mathbf{e}_2$ as $\alpha_1 \mathbf{e}_1 + \beta \mathbf{e}_3$, we get:

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \alpha \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ \beta \end{bmatrix},$$

which implies $1 = 0$, a contradiction.

Thus, $\mathbf{e}_2$ cannot be written as a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_3$.

Properties of the standard basis

(ii) If we have a linear combination

$$\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \cdots + \lambda_n \mathbf{e}_n = \mathbf{0},$$

then $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

For example:. If $\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 = \mathbf{0}$, then:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix},$$

so $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Properties of the standard basis

(iii) Any vector in $\mathbb{R}^n = \text{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ can be written **uniquely** as a linear combination of $\mathbf{e}_1, \ldots, \mathbf{e}_n$.

In other words: If

$$\alpha_1 \mathbf{e}_1 + \cdots + \alpha_n \mathbf{e}_n = \beta_1 \mathbf{e}_1 + \cdots + \beta_n \mathbf{e}_n,$$

Then:

$$\alpha_1 = \beta_1, \ldots, \alpha_n = \beta_n.$$

For example: If

$$\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 = \beta \mathbf{e}_1 + \beta_2 \mathbf{e}_2 + \beta_3 \mathbf{e}_3,$$

Then:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 = \beta \mathbf{e}_1 + \beta_2 \mathbf{e}_2 + \beta_3 \mathbf{e}_3 = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix},$$

so, $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \alpha_3 = \beta_3$.

Theorem

Very Important Theorem. Let $v_1, \ldots, v_r$ be vectors in $\mathbb{R}^n$ and let A denote the $n \times r$ matrix $A = [v_1 \;\; v_2 \;\; \cdots \;\; v_r]$. The following equivalent conditions are equivalent::

(i) No vector in the list can be written as a linear combination of the remaining vectors in the list.

(ii) If we have a linear combination

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_r v_r = \mathbf{0},$$

then $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$.

(iii) Any vector in $\operatorname{span}\{v_1, \ldots, v_r\}$ can be written **uniquely** as a linear combination of $v_1, \ldots, v_r$.

(iv) The system of equations $A \cdot \mathrm{X} = \mathbf{0}$ has only the 0 solution.

Note that condition (iv) is equivalent to the conditions (i)-(iii) since $A \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{bmatrix} = \mathbf{0}$ gives a solution to (iv) if and only if $\lambda_1 v_1 + \cdots \lambda_r v_r = \mathbf{0}$.

Definition

Important Definition:

Vectors $v_1, \ldots, v_r$ in $\mathbb{R}^n$ are said to be linearly independent if any of the four equivalent conditions in the theorem above hold.

Example

Example: Verify that the vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are linearly independent.

Solution:We must show that for $A = [v_1 \ v_2 \ v_3]$, the system $A \cdot \mathrm{X} = \mathbf{0}$ has only the zero solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_2+R_3]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \xrightarrow[\frac{1}{2}\cdot R_3]{-1\cdot R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

which shows that the system $A \cdot \mathrm{X} = \mathbf{0}$ has a unique solution.Therefore, the vectors v_1, v_2, v_3 are linearly independent.