Lecture 15: Subspaces of Euclidean Space and Spanning Sets

Examples of subspaces of Euclidean Space

Recall that Euclidean n-space, denoted $\mathbb{R}^n$, is the set of all n-tuples of real numbers of length n. Such vectors can be written either as row vectors $(a_1, a_2, \ldots, a_n)$ or column vectors $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$.

We add and scalar multiple coordinate wise:Thus,

$$2 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 5 \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 20 \\ 25 \\ 30 \end{bmatrix} = \begin{bmatrix} 22 \\ 29 \\ 36 \end{bmatrix}.$$

Recall that if $v_1, \ldots v_r \in \mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, then

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_r v_r$$

is a linear combination of $v_1, \ldots, v_r$.

Examples of subspaces of Euclidean Space

1. Let

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ \vdots \qquad\quad &= \vdots \\ a_{m1} + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

be a homogeneous system of m linear equations in n unknowns. Recall that if $v = (s_1, \ldots, s_n)$ and $u = (t_1, \ldots, t_n)$ are solutions to the system, then:

(i) (0,. . . , 0) is a solution.

(ii) $v + u$ is a solution.

(iii) λv is a solution, for all $\lambda \in \mathbb{R}$.

In particular : If $v_1, \ldots, v_r$ are solutions, then any linear combination

$$\lambda_1 v_1 + \cdots + \lambda_r v_r$$

is a solution to the homogeneous system.

Examples of subspaces of Euclidean Space

Let A be an $m \times n$ matrix. We can think of multiplication by A as a function from $\mathbb{R}^n$ to $\mathbb{R}^m$, since $v \in \mathbb{R}^n$ implies $Av \in \mathbb{R}^m$.

2. Let W denote the **nullspace of A**, i.e., the set of all $v \in \mathbb{R}^n$ such that $Av = \mathbf{0}$. Then :

(i) $\mathbf{0} \in W$

(ii) If $v_1, v_2 \in W$, then $v_1 + v_2 \in W$, since:

$$A(v_1 + v_2) = Av_1 + Av_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

(iii) If $v \in W$, then $\lambda v \in W$, for all $\lambda \in \mathbb{R}$, since:

$$A(\lambda v) = \lambda Av = \lambda \mathbf{0} = \mathbf{0}.$$

In particular: If $v_1, \ldots, v_r$ are in W then any linear combination

$$\lambda_1 v_1 + \cdots + \lambda_r v_r$$

is also in W.

Examples of subspaces of Euclidean Space

3. Let $U \subseteq \mathbb{R}^m$ denote the **image of A**. That is, all vectors in $\mathbb{R}^m$ of the form Av, with $v \in \mathbb{R}^n$. Then:

(i) $\mathbf{0} \in U$, since $\mathbf{0}_{\mathbb{R}^m} = A \cdot \mathbf{0}_{\mathbb{R}^n}$.

(ii) If Av_1, Av_2 are in U, then $Av_1 + Av_2 = A(v_1 + v_2)$ is in U.

(iii) If Av is in U, then $\lambda(Av) = A(\lambda v)$ is in U, for all $\lambda \in \mathbb{R}$.

In particular: If $v_1, \ldots, v_r$ are in U then any linear combination

$$\lambda_1 v_1 + \cdots + \lambda_r v_r$$

is also in U.

Examples of subspaces of Euclidean Space

4. Now assume that A is an $n \times n$ matrix and let λ be an eigenvalue.

Let $E_\lambda(A) \subseteq \mathbb{R}^n$ denote the set of all λ-eigenvectors of A, together with the zero vector $\mathbf{0}$ of $\mathbb{R}^n$. $E_\lambda(A)$ is called the **eigenspace** of λ. Then:

(i) $\mathbf{0} \in E_\lambda(A)$, by definition.

(ii) If v_1, v_2 are in $E_\lambda(A)$, then $v_1 + v_2$ is in $E_\lambda(A)$, since:

$$A(v_1 + v_2) = Av_1 + Av_2 = \lambda v_1 + \lambda v_2 = \lambda(v_1 + v_2).$$

(iii) If v is in $E_\lambda(A)$, then γv is in $E_\lambda(A)$, for all $\gamma \in \mathbb{R}$, since:

$$A(\gamma v) = \gamma A v = \gamma(\lambda v) = \gamma\lambda v = \lambda(\gamma v).$$

In particular: If $v_1, \ldots, v_r$ are in $E_\lambda(A)$ then any linear combination

$$\gamma_1 v_1 + \cdots + \gamma_r v_r$$

is also in $E_\lambda(A)$.

Examples of subspaces of Euclidean Space

5. Fix vectors $v_1, \ldots, v_r \in \mathbb{R}^n$. The **span** of $v_1, \ldots, v_r$ is the set of all linear combinations of $v_1, \ldots, v_r$.

Let S denote the span of the vectors $v_1, \ldots, v_r$.

(i) $\mathbf{0} \in S$, since $\mathbf{0} = 0 \cdot v_1 + \cdots + 0 \cdot v_r$.

(ii) If $v = \alpha_1 v_1 + \cdots + \alpha_r v_r \in S$ and $u = \beta_1 v_1 + \cdots + \beta_r v_r \in S$, then $v + u \in S$, since:

$$\begin{aligned} v + u &= (\alpha_1 v_1 + \cdots + \alpha_r v_r) + (\beta_1 v_1 + \cdots + \beta_r v_r) \\ &= (\alpha_1 + \beta_1) v_1 + \cdots + (\alpha_r + \beta_r) v_r. \end{aligned}$$

(iii) If $v = \alpha_1 v_1 + \cdots + \alpha_r v_r \in S$ and $\gamma \in \mathbb{R}$, then $\gamma v \in S$, since:

$$\begin{aligned} \gamma v &= \gamma(\alpha_1 v_1 + \cdots + \alpha_r v_r \in S) \\ &= (\gamma \alpha_1) v_1 + \cdots + (\gamma \alpha_r) v_r \end{aligned}$$

Examples of subspaces of Euclidean Space

In particular: If $w_1, \ldots, w_t$ are in S then any linear combination

$$\gamma_1 w_1 + \cdots + \gamma_t w_t$$

is also in S.

In other words: A linear combination of linear combinations of $v_1, \ldots, v_r$ is a linear combination of $v_1, \ldots, v_r$.

Definition

A subset $W \subseteq \mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if it satisfies the following properties:

(i) $\mathbf{0} \in W$.

(ii) If $v_1, v_2 \in W$, then $v_1 + v_2 \in W$.

(iii) if $v \in W$, then $\lambda v \in W$, for all $\lambda \in \mathbb{R}$.

In particular: If $v_1, \ldots, v_r$ are in W then any linear combination

$$\gamma_1 v_1 + \cdots + \gamma_r v_r$$

is also in W.

Therefore: The Examples 1-5 above are all subspaces of $\mathbb{R}^n$.

Example

What are the subspaces of $\mathbb{R}^2$?

(i) $\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^2$.

(ii) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

(iii) Any line through the origin is a subspace of $\mathbb{R}^2$.

These are the only subspaces of $\mathbb{R}^2$.

Let L be a line in $\mathbb{R}^2$ through the origin. L is given by $y = mx$, where m is the slope of L.

Thus, L is the solution set of the homogeneous equation $x + (-m)y = 0$, which shows that L is a subspace.

Example

The subspaces of $\mathbb{R}^3$ are:

(i) $\{\mathbf{0}\}$.

(ii) $\mathbb{R}^3$.

(iii) Any line through the origin.

(iv) Any plane through the origin.

Why: Each of these sets are the solution set to a homogeneous system of equations. For example:

(a) Any plane through the origin is defined by an equation of the form: $ax + by + cz = 0$, for fixed constants $a, b, c \in \mathbb{R}$.

(b) Any line through the origin is the intersection of two planes through the origin and is thus the solution set to a system of equations of the form:

$$\begin{aligned} ax + by + cz &= 0 \\ dx + ey + fz &= 0. \end{aligned}$$

Example

A line or plane in $\mathbb{R}^2$ or $\mathbb{R}^3$ that does not pass through the origin, is NOT a subspace. We require subspaces to contain $\mathbf{0}$.

Another reason: Suppose $L : y = mx + b$ is a line in $\mathbb{R}^2$, with $b \neq 0$. Let $v_1 = \begin{bmatrix} a \\ ma + b \end{bmatrix}$ and $v_2 = \begin{bmatrix} c \\ mc + b \end{bmatrix}$ be vectors in L. Then $v_1 + v_2 = \begin{bmatrix} a + c \\ m(a + c) + 2b \end{bmatrix}$, which does not belong to L.

Definition

Let $U \subseteq \mathbb{R}^n$ be a subspace. Vectors $v_1, \ldots, v_r \in U$ span U if U is the subspace spanned by $v_1, \ldots, v_r$,

In other words, U is the set of all linear combinations of $v_1, \ldots, v_r$.

In this case we write $U = \text{span}\{v_1, \ldots, v_r\}$.

Important Observation: Let $A = [v_1 \ v_2 \ \cdots \ v_r]$ be the $n \times r$ matrix whose columns are $v_1, \ldots, v_r$. Then:

$$u \in U \quad \text{if and only if} \quad u = A \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{bmatrix},$$

since

$$A \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{bmatrix} = [v_1 \cdots v_r] \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{bmatrix} = \lambda_1 v_1 + \cdots + \lambda_r v_r.$$

Comment

Thus: A vector $b \in \mathbb{R}^n$ is in the span of $v_1, \ldots, v_r$ if and only if the $n \times r$ system of equations given by

$$A \cdot \mathbf{X} = b$$

has a solution.

Moreover, if $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{bmatrix}$ is a solution, then

$$b = \lambda_1 v_1 + \cdots + \lambda_r v_r.$$

Important Consequence: Given any $m \times n$ matrix A, the subspace $\mathrm{im}(A)$, the image of A, is the subspace spanned of $\mathbb{R}^m$ spanned by the columns of A.

Example

Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Is $b = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ in $\text{span}\{v_1, v_2, v_3\}$?

Solution: Let $A = [v_1 \ v_2 \ v_3]$. We seek a solution to $A \cdot \mathbf{X} = b$. We use Gaussian elimination.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-1 \cdot R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{array}\right] \xrightarrow{-1 \cdot R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2} \cdot R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \xrightarrow[-1 \cdot R_3 + R_2]{-1 \cdot R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right].$$

This shows that $b = \frac{3}{2} \cdot v_1 - \frac{1}{2} \cdot v_2 + \frac{1}{2} \cdot v_3.$,

Thus, $b \in \text{span}\{v_1, v_2, v_3\}$.

Example continued

CHECK: $\frac{3}{2} \cdot v_1 - \frac{1}{2} \cdot v_2 + \frac{1}{2} \cdot v_3 =$

$$\frac{3}{2} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ \frac{3}{2} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = b.$$

Class Example

Determine if $b = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$ belongs to span$\{v_1, v_2, v_3\}$, for

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

If so, write b as a linear combination of v_1, v_2, v_3.

Then, verify your answer.

Class Example continued

Solution: Using Gaussian Elimination,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 1 & 1 & 3 \\ 1 & 1 & 2 & 1 \end{array}\right] \xrightarrow[-1\cdot R_1+R_3]{-2\cdot R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -3 & -1 & -5 \\ 0 & -1 & 1 & -3 \end{array}\right] \xrightarrow{R_2\leftrightarrow -R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & -3 & -1 & -5 \end{array}\right]$$

$$\xrightarrow[3\cdot R_2+R_3]{-2\cdot R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -4 & 4 \end{array}\right] \xrightarrow{-\frac{1}{4}\cdot R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[1\cdot R_3+R_2]{-3\cdot R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Thus: $b = 1 \cdot v_1 + 2 \cdot v_2 - v_3$.

Class Example continued

CHECK: $1 \cdot v_1 + 2 \cdot v_2 - v_3 =$

$$1 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$