Lecture 14: Applications of Diagonalization Continued

First Application: Solving recurrence relations.

The sequence of non-negative numbers $a_0, a_1, a_2, \ldots, a_k, \ldots,$ is called a linear recursion sequence of length two if there are fixed integers α, β, c, d such that:

(i) $a_0 = \alpha$.

(ii) $a_1 = \beta$.

(iii) $a_{k+2} = c \cdot a_k + d \cdot a_{k+1}$, for all $k \geq 0$.

The conditions in (i) and (ii) are called *initial conditions.*

To solve the recurrence relation, we set up a matrix equation. Let $v_k = \begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix}$, and $A = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}$. Then $Av_k = v_{k+1}$, for all $k \geq 0$.

Therefore $v_k = A^k \cdot v_0$.

If $A = PDP^{-1}$. with D diagonal, compute A^k as PD^kP^{-1}.

Solution: Then a_k is the first coordinate of the vector

$$A^k \cdot v_0 = PD^nP^{-1} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Applications

Second Application: Systems of First Order Linear Differential Equations.

Let $A = (a_{ij})$, be an $n \times n$ matrix. A system of first order linear differential equations is a system of equations of the form:

$$\begin{aligned} x_1'(t) &= a_{11}x_1(t) + \cdots + a_{1n}x_n(t) \\ x_2'(t) &= a_{21}x_1(t) + \cdots + a_{2n}x_n(t) \\ \vdots \quad &= \qquad\qquad \vdots \\ x_n'(t) &= a_{n1}x_1(t) + \cdots + a_{nn}x_n(t), \end{aligned}$$

where $x_i(t)$ is a real valued function of t. The numbers $x_1(0), \cdots, x_n(0)$ are called the *initial conditions* of the system.

Applications

To solve the system, we convert to a single vector valued first order linear differential equation as follows:

Set $\mathbf{X}(t) := \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$. Then the system above simply becomes:

$$\mathbf{X}(t) = A \cdot \mathbf{X}'(t)$$

The fixed vector $\mathbf{X}(0)$ represents the initial condition.

The solution to the vector equation is:

$$\mathbf{X(t) = e^{At} \cdot X(0)}$$

One then converts this vector equation back to individual solutions for $x_i(t)$, for all i.

Example

Find the solution to the system of first order linear differential equations:

$$
\begin{aligned}
x_1'(t) &= x_1(t) + x_2(t) + x_3(t) \\
x_2'(t) &= x_1(t) + x_2(t) + x_3(t) \\
x_3'(t) &= x_1(t) + x_2(t) + x_3(t).
\end{aligned}
$$

with initial conditions $x_1(0) = -1, x_2(0) = 1, x_3(0) = 2$.

Solution: The coefficient matrix for the system is $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. So the solution to the system is $e^{At} \cdot \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

Example continued

To calculate, e^{At}, we must diagonalize A. By direct calculation, or using linear algebra software, we have $c_A(x) = x^3 - 3x^2$, so that the eigenvalues are 0, with multiplicity two, and 3 with multiplicity one. .

The basic eigenvectors of 0, are just the solution space of $0I_3 - A = -A$, which reduces to $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore, basic 0-eigenvectors are: $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

Example continued

Using linear algebra software, a basic 3-eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Thus, $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$ is the matrix of eigenvectors.

Linear algebra software yields: $P^{-1} = \frac{1}{3} \cdot \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$.

Example continued

Now, $A = PDP^{-1}$, for

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{so} \quad Dt = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3t \end{bmatrix}. \text{ Thus, } e^{Dt} = \begin{bmatrix} e^0 & 0 & 0 \\ 0 & e^0 & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$$

Then, $e^{At} = Pe^{Dt}P^{-1} =$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{3t} \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ e^{3t} & e^{3t} & e^{3t} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 + e^{3t} & -1 + e^{3t} & -1 + e^{3t} \\ -1 + e^{3t} & 2 + e^{3t} & -1 + e^{3t} \\ -1 + e^{3t} & -1 + e^{3t} & 2 + e^{3t} \end{bmatrix}.$$

Example continued

Finally,

$$\mathbf{X}(t) = e^{At} \cdot \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2+e^{3t} & -1+e^{3t} & -1+e^{3t} \\ -1+e^{3t} & 2+e^{3t} & -1+e^{3t} \\ -1+e^{3t} & -1+e^{3t} & 2+e^{3t} \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} -5+2e^{3t} \\ 1+2e^{3t} \\ 4+2e^{3t} \end{bmatrix}.$$

Therefore,

$$x_1(t) = -\frac{5}{3} + \frac{2}{3}e^{3t},\ x_2(t) = \frac{1}{3} + \frac{2}{3}e^{3t},\ x_3(t) = \frac{4}{3} + \frac{2}{3}e^{3t}.$$

Example continued

Check: On the one hand,

$$x_1'(t) = 2e^{3t},\ x_2'(t) = 2e^{3t},\ x_3'(t) = 2e^{3t}.$$

One the other hand,

$$\begin{aligned} x_1(t) + x_2(t) + x_3(t) &= (-\frac{5}{3} + \frac{2}{3}e^{3t}) + (\frac{1}{3} + \frac{2}{3}e^{3t}) + (\frac{4}{3} + \frac{2}{3}e^{3t}) \\ &= 2e^{3t}. \end{aligned}$$

Thus:

$$\begin{aligned} x_1'(t) &= x_1(t) + x_2(2) + x_3(t) \\ x_2'(t) &= x_1(t) + x_2(t) + x_3(t) \\ x_3'(t) &= x_1(t) + x_2(t) + x_3(t) \end{aligned}$$

as required.

Example continued

To check the initial conditions:

$$x_1(0) = -\frac{5}{3} + \frac{2}{3}e^0 = -1$$
$$x_2(0) = \frac{1}{3} + \frac{2}{3}e^0 = 1$$
$$x_3(0) = \frac{4}{3} + \frac{2}{3}e^0 = 2.$$

Class Example

Find the solution to the system of linear first order differential equations with given initial conditions:

$$\begin{aligned} x_1'(t) &= -22x_1(t) + 50x_2(t) \\ x_2'(t) &= -10x_1(t) + 23x_2(t), \end{aligned}$$

and $x_1(0) = -3$ and $x_2(0) = 2$.

Verify your solution satisfies the system and the initial conditions.

Solution: $\mathbf{X}(t) = \begin{bmatrix} x_t(t) \\ x_2(t) \end{bmatrix} = e^{At} \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, for $A = \begin{bmatrix} -22 & 50 \\ -10 & 23 \end{bmatrix}$.
$c_A(x) = x^2 - x - 6 = (x+2)(x-3)$.

The eigenvalues are: -2, 3 with eigenvectors $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Take $P = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$, so $P^{-1} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$.

Class Example continued

$e^{At} = Pe^{Dt}P^{-1}$, where $Dt = \begin{bmatrix} -2t & 0 \\ 0 & 3t \end{bmatrix}$. So, $e^{Dt} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{3t} \end{bmatrix}$.

Therefore:

$$e^{At} = Pe^{Dt}P^{-1} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-2t} & -2e^{-2t} \\ -2e^{3t} & 5e^{3t} \end{bmatrix}$$

$$= \begin{bmatrix} 5e^{-2t} + -4e^{3t} & -10e^{-2t} + 10e^{3t} \\ 2e^{-2t} - 2e^{3t} & -4e^{-2t} + 5e^{3t} \end{bmatrix}.$$

Class Example continued

Therefore:

$$\mathbf{X}(t) = \begin{bmatrix} x_t(t) \\ x_2(t) \end{bmatrix} = e^{At} \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 5e^{-2t} + -4e^{3t} & -10e^{-2t} + 10e^{3t} \\ 2e^{-2t} - 2e^{3t} & -4e^{-2t} + 5e^{3t} \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -35e^{-2t} + 32e^{3t} \\ -14e^{-2t} + 16e^{3t} \end{bmatrix}.$$

Thus:

$$x_1(t) = -35e^{-2t} + 32e^{3t}$$
$$x_2(t) = -14e^{-2t} + 16e^{3t}.$$

Class Example continued

CHECK: On the one hand. $x_1'(t) = 70e^{-2t} + 96e^{3t}$. On the other hand:

$$\begin{aligned} -22x_1(t) + 50x_2(t) &= -22(-35e^{-2t} + 32e^{3t}) + 50(-14e^{2t} + 16e^{3t}) \\ &= 70e^{-2t} + 96e^{3t} \end{aligned}$$

And: $x_2'(t) = 28e^{-2t} + 48e^{3t}$, while:

$$\begin{aligned} -10x_1(t) + 23x_2(t) &= -10(-35e^{-2t} + 32e^{3t}) + 23(-14e^{-2t} + 16e^{3t}) \\ &= 28e^{-2t} + 48e^{3t}. \end{aligned}$$

For the initial conditions:

$$\begin{aligned} x_1(0) &= -35e^0 + 32e^0 = 35 - 32 = -3 \\ x_2(0) &= -14e^0 + 16e^0 = -14 + 16 = 2, \end{aligned}$$

as required.