Lecture 12: Eigenvalues, Eigenvectors and Diagonalization, continued

Eigenvectors and Eigenvalues

Let A be an $n \times n$ matrix. The real number λ is called an **eigenvalue** of A if there exists a **non-zero** vector $v \in \mathbb{R}^2$ such that $Av = \lambda v$.

The vector v is called an eigenvector of A associated to λ or a λ-**eigenvector**.

To find the eigenvalues of the $n \times n$ matrix A:

$$\text{Solve the equation } |\lambda \cdot I_n - A| = 0, \text{ for } \lambda.$$

To find the λ-eigenvectors of the $n \times n$ matrix A: Find the solutions to the homogenous matrix equation

$$(\lambda I_n - A) \cdot \vec{X} = \vec{0}.$$

The basic solutions to this homogeneous system are called **basic eigenvectors for λ** or **basic λ-eigenvectors**.

Thus, every λ-eigenvector is a linear combination of basic λ-eigenvectors.

Definition

Let A be an $n \times n$ matrix.

(i) The polynomial $c_A(x) = |x \cdot I_n - A|$ is called the **characteristic polynomial of A**. For any real number λ, $c_A(\lambda) = |\lambda \cdot I - A|$, so that λ is an eigenvalue of A if and only if λ is a root of $c_A(x)$.

(ii) The eigenvalue λ has **multiplicity m**, if λ occurs m times as a root of $c_A(x)$.

(iii) The matrix A is **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Example

Let $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find the characteristic polynomial of C, the eigenvalues (with multiplicities), the eigenvectors and a diagonalizing matrix.

The characteristic polynomial of C is:

$$c_C(x) = |x \cdot I - C| = \begin{vmatrix} x-1 & -1 & 0 \\ 0 & x-2 & 0 \\ 0 & 0 & x-1 \end{vmatrix} = (x-1)^2(x-2).$$

Thus , the eigenvalues of C are 1, 1, 2, and 1 is eigenvalue of multiplicity two. From last lecture, we saw that the basic eigenvectors for 1 are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Example continued

The eigenvector for 2 is given by the nullspace of the matrix

$$2I_2 - C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is a basic eigenvector for 2.Now put all three basic eigenvectors in to a matrix P, so that $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Calculation (or computer software) gives $P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Example continued

Calculating $P^{-1}CP$ we get

$$\begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\cdot\begin{bmatrix}1 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}=\begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 2\\ 0 & 0 & 2\\ 0 & 1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}.$$

Class Example

For the matrix $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, find $c_A(x)$, the eigenvalues and eigenvectors, then diagonalize A.

Solution: $c_A(x) = |xI_2 - a| = \begin{vmatrix} x+1 & -2 \\ 0 & x-1 \end{vmatrix} = x^2 - 1 = (x-1)(x+1)$.
Therefore the eigenvalues of A are 1, -1.

To find the -1-eigenvectors:

$$-1 \cdot I_2 - A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & -2 \end{bmatrix} \xrightarrow{-1 \cdot R_1 + R_2} \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix},$$

so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a basic eigenvector for 1.

Class Example continued

To find the 1-eigenvectors:

$$1 \cdot I_2 - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix},$$

so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basic vector for 1. Set $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.
Thus, $P^{-1}AP =$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

When is a matrix diagonalizable?

Theorem. Let A be an $n \times n$ matrix. The following conditions are equivalent.

(i) A is diagonalizable

(ii) $c_A(x) = (x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2} \cdots (x - \lambda_r)^{m_r}$ and for each λ_i, A has m_i basic vectors.

Moreover: When this is the case, if $v_1, \ldots, v_n$ are the n basic vectors from (ii), and we let P denote the $n \times n$ matrix whose columns are the v_i, then $P^{-1}AP$ is the $n \times n$ matrix with

$$\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_r, \ldots, \lambda_r$$

down its main diagonal, where each λ_i appears m_i times.

To summarize: The $n \times n$ matrix A is diagonalizable, if A has n eigenvalues (counted with multiplicities) and for each eigenvalue λ, if the multiplicity of λ is m, then A must have m basic eigenvectors.

Illustrative Examples Revisited

We apply the theorem to each of the Illustrative Examples from the last lecture.

Example 1. **No eigenvalues.** For $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,

$c_A(x) = |\lambda I_2 - A| = \begin{vmatrix} x & 1 \\ -1 & x \end{vmatrix} = x^2 + 1$, which has no real roots, and thus A no eigenvalues (over $\mathbb{R}$). Therefore, A is not diagonalizable. (Recall from last lecture, A is a rotation matrix.)

Example 2. **An eigenvalue with multiplicity two and one basic eigenvector.** Consider $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then

$$c_B(x) = |xI_2 - B| = \begin{vmatrix} x-1 & -1 \\ 0 & x-1 \end{vmatrix} = (x-1)^2,$$

so 1 is an eigenvalue of multiplicity two.

On the other hand, we saw that B has just one basic eigenrvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Therefore B is not diagonalizable

Illustrative Examples Revisited

Example 3. A diagonalizable matrix with an eigenvalue of multiplicity greater then one. For the matrix C in the example above, 1 is an eigenvalue of multiplicity two, with two basic eigenvectors and 2 is an eigenvalue of multiplicity one, with one basic eigenvector.

C is diagonalizable and the diagonalizing matrix P is obtained by taking the basic eigenvectors as columns.

Example 4. A eigenvalue of multiplicity three and one basic eigenvector. Consider $D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$$c_D(x) = |\lambda \cdot I_3 - D| = \begin{vmatrix} x-1 & -1 & 0 \\ 0 & x-1 & -1 \\ 0 & 0 & x-1 \end{vmatrix} = (x-1)^3,$$

so 1 is an eigenvalue of D of multiplicity three.

Moreover, we saw that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the only basic vector, so D is not diagonalizable.

Comment

Computing powers of a diagonalizable matrix: Suppose A is diagonalizable. We want to compute A^n, all n. Then $P^{-1}AP = D$, where $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Note that $D^r = \text{diag}(\lambda_1^r, \ldots, \lambda_n^r)$, for all r.

To compute the powers of A, we note that $A = PDP^{-1}$.

(i) $A^2 = PDP^{-1} \cdot PDP^{-1} = PD^2P^{-1}$.

(ii) $A^3 = A^2 \cdot A = PD^2P^{-1} \cdot PDP^{-1} = PD^3P^{-1}$.

(iii) Continuing, $A^n = PD^nP^{-1}$, for all n.

Thus, if A is diagonalizable, in order to calculate the powers of A, we just have to diagonalize A and compute the powers of a diagonal matrix.

Example

From an earlier example, for $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, A has eigenvalues $-1, 1$ with eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus, we take $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $P^{-1}AP = D$, and thus $A^n = PD^nP^{-1}$, for $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore, $D^n = \begin{bmatrix} (-1)^n & 0 \\ 0 & 1 \end{bmatrix}$. Thus,

$$A^n = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} (-1)^n & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} (-1)^n & (-1)^{n+1} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)^n & (-1)^{(n+1)} + 1 \\ 0 & 1 \end{bmatrix}$$

Applications

First Application: Solving recurrence relations.

Consider the sequence of non-negative numbers $a_0, a_1, a_2, \ldots$, where

$$a_0 = 1, a_1 = 2, \ldots, a_{k+1} = -7a_k + 8a_{k-1},$$

for $k \geq 1$. Thus, $a_2 = -7 \cdot 2 + 8 \cdot 1 = -6, a_3 = -7 \cdot (-6) + 8 \cdot 2 = 58$.

Find the value of a_k for all $k \geq 0$.

Solution: Set up a matrix equation. Let $v_k = \begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix}$, and $A = \begin{bmatrix} 0 & 1 \\ 8 & -7 \end{bmatrix}$.
Thus, for $k \geq 0$,

$$A \cdot v_k = \begin{bmatrix} 0 & 1 \\ 8 & -7 \end{bmatrix} \cdot \begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} a_{k+1} \\ a_{k+2} \end{bmatrix} = v_{k+1}.$$

Since $v_1 = Av_0$ and $v_2 = Av_1$, we have $v_2 = A^2 v_0$. And:
$v_3 = Av_2 = A \cdot A^2 v = A^3 v_0$. Continuing, we have $v_k = A^k v_0$, for all k.

Applications Continued

Thus: To find v_k, we must calculate A^k, and for this, we will diagonalize A. $c_A(x) = \begin{vmatrix} x & -1 \\ -8 & x+7 \end{vmatrix} = x^2 + 7x - 8$. Therefore, the eigenvalues of A are $1, -8$.

The usual calculation leads to: The basic eigenvector for 1 is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the basic eigenvector for -8 is $\begin{bmatrix} 1 \\ -8 \end{bmatrix}$. Now we take $P = \begin{bmatrix} 1 & 1 \\ 1 & -8 \end{bmatrix}$. Thus,

$P^{-1} = \begin{bmatrix} \frac{8}{9} & \frac{1}{9} \\ \frac{1}{9} & -\frac{1}{9} \end{bmatrix}$. From before, we have $A^k = PD^kP^{-1}$, where $D = \begin{bmatrix} 1 & 0 \\ 0 & -8 \end{bmatrix}$. Thus,

Applications Continued

$$A^k = \begin{bmatrix} 1 & 1 \\ 1 & -8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & (-8)^k \end{bmatrix} \cdot \begin{bmatrix} \frac{8}{9} & \frac{1}{9} \\ \frac{1}{9} & -\frac{1}{9} \end{bmatrix} = \begin{bmatrix} 1 & (-8)^k \\ 1 & (-8)^{k+1} \end{bmatrix} \cdot \begin{bmatrix} \frac{8}{9} & \frac{1}{9} \\ \frac{1}{9} & -\frac{1}{9} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8+(-8)^k}{9} & \frac{1-(-8)^k}{9} \\ \frac{8+(-8)^{k+1}}{9} & \frac{1-(-8)^{k+1}}{9} \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix} = v_k = A^k \cdot v_0 = \begin{bmatrix} \frac{8+(-8)^k}{9} & \frac{1-(-8)^k}{9} \\ \frac{8+(-8)^{k+1}}{9} & \frac{1-(-8)^{k+1}}{9} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{8+(-8)^k+2-2(-8)^k}{9} \\ \text{—} \end{bmatrix}.$$

Thus, $a_k = \frac{8+(-8)^k+2-2(-8)^k}{9} = \frac{10-(-8)^k}{9}$.

Definition

Let A be an $n \times n$ matrix and e, the Euler number. Define e^A by the formula

$$e^A = I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \cdots$$

Applications Continued

Calculating e^A for A diagonalizable.

Suppose A is diagonalizable. Then $A = PDP^{-1}$ for D an $n \times n$ diagonal matrix with the eigenvalues of A down its main diagonal.

Thus, $A^n = PD^nP^{-1}$, for all n, as before.Therefore:

$$\begin{aligned} e^A &= I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \\ &= I_n + (PDP^{-1}) + \frac{1}{2!}(PD^2P^{-1}) + \frac{1}{3!}(PD^3P^{-1}) + \cdots \\ &= P\{I_n + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots\}P^{-1} \\ &= Pe^DP^{-1}. \end{aligned}$$

Applications Continued

To calculate e^D, suppose $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Then

$$\frac{1}{r!}D^r = \mathrm{diag}(\frac{\lambda_1^r}{r!}, \ldots, \frac{\lambda_n^r}{r!}).$$

Summing from r 0 to ∞, we see

$$e^D = \sum_{i=0}^{\infty} \frac{1}{r!}D^r = \sum_{i=0}^{\infty} \mathrm{diag}(\frac{\lambda_1^r}{r!}, \ldots, \frac{\lambda_n^r}{r!}) = \text{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n}).$$

For example: if $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, then we have seen that $A = PDP^{-1}$, for $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus:

$$e^A = Pe^D P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-1} & 0 \\ 0 & e \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{-1} & e \\ 0 & e \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-1} & e - e^{-1} \\ 0 & e \end{bmatrix}.$$