Lecture 11: Eigenvalues, Eigenvectors and Diagonalization

Eigenvectors and Eigenvalues

Let A be an $n \times n$ matrix. The real number λ is called an **eigenvalue** of A if there exists a **non-zero** vector $v \in \mathbb{R}^2$ such that $Av = \lambda v$.

The vector v is called an eigenvector of A associated to λ or a λ-**eigenvector**.

Example: Let $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$, $v = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. Then

$$A \cdot v = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 4 \end{bmatrix} = 4 \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 4 \cdot v,$$

so 4 is an eigenvalue of A with eigenvector v.

Example continued

For the matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$, -2 is a second eigenvalue with associated eigenvalue $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Comment

If λ is an eigenvalue for the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with eigenvector v, then $Av = \lambda v$.

On the other hand, $\lambda v = (\lambda I_2) \cdot v = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \cdot v$

Thus, $Av = (\lambda I_2)v$, so $(\lambda I_2 - A)v = 0$.

Therefore, the homogeneous system of equations

$$(\lambda I_2 - A)X = \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix} \cdot X = 0$$

has a non-trivial solution.

Comment

Consequently: If λ is an eigenvalue of A with eigenvector v, then:

(i) $\det(\lambda I_2 - A) = \det(\begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix}) = 0.$

(ii) v is in the nullspace of $(\lambda I_2 - A)$.

In fact: Items (i) and (ii) hold for any $n \times n$ matrix. in other words

To find the eigenvalues of the $n \times n$ matrix A:

$$\text{Solve the equation } |\lambda \cdot I_n - A| = 0, \text{ for } \lambda.$$

To find the λ-eigenvectors of the $n \times n$ matrix A: Find the basic solutions to the homogenous matrix equation

$$(\lambda I_n - A) \cdot \vec{X} = \vec{0}.$$

Example

Find the eigenvectors and eigenvalues for $A = \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix}$.

Solution: Set $\det(\lambda I_2 - A) = \begin{vmatrix} \lambda - 0 & 6 \\ -1 & \lambda - 5 \end{vmatrix} = 0$. Thus,

$$\lambda(\lambda - 5) + 6 = \lambda^2 - 5\lambda + 6 = 0.$$

Therefore $\lambda = 2$ and $\lambda = 3$ are the eigenvalues of A.

Example continued

To find an eigenvector for 2 we need the nullspace of

$$(2I_2 - A) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}.$$

Using EROs:

$$\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} \xrightarrow{\frac{1}{2}\cdot R_1} \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix} \xrightarrow{1\cdot R_1 + R_2} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix},$$

from which we see that the nullspace is generated by $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

In other words, $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is a 2-eigenvector of A.

In fact, all of the vectors $s \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, with $s \in \mathbb{R}$ are 2-eigenvectors of A.

Example continued

To find an eigenvector for 3 we need the nullspace of

$$(3I_2 - A) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}.$$

Using EROs:

$$\begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix} \xrightarrow{\frac{1}{3}\cdot R_1} \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \xrightarrow{1\cdot R_1 + R_2} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix},$$

from which we see that the nullspace is generated by $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

In other words, $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is a 3-eigenvector of A.

In fact, all of the vectors $r \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, with $r \in \mathbb{R}$ are 2-eigenvectors of A.

Example

Find the eigenvalues and eigenvectors for $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Solution:

$$|\lambda I_3 - A| = \begin{vmatrix} \lambda & -1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2) \cdot \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = (\lambda - 2)(\lambda^2 - 1).$$

Thus, the eigenvalues of A are: 2, 1, -1.

Example continued

To find the 2-eigenvectors, we find the solutions to the equation $(2I_3 - A) \cdot \vec{X} = \vec{0}$:

$$2I_3 - A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2} \cdot R_1 + R_2} \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This shows that $2I_3 - A$ has rank two, and thus, one basic solution: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Therefore the vectors $\{s \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$ with $s \in \mathbb{R}$ are all of the 2-eigenvectors.

Example continued

For the 1-eigenvectors:

$$1 \cdot I_3 - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{1 \cdot R_1 + R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

This shows that $1 \cdot I_3 - A$ has rank two, and there is one basic solution: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Example continued

For the -1-eigenvectors:

$$-1 \cdot I_3 - A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\xrightarrow{-1\cdot R_1 + R_2} \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

This shows that $-1 \cdot I_3 - A$ has rank two and thus, there is one basic solution: $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Thus, for this example, there are three distinct eigenvalues, each with a single eigenvector associated to it.

We will see below, that this phenomenon need not occur.

Class Example

For the matrices $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -6 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}$, find the eigenvalues and eigenvectors for A, and find the eigenvalues for B.

Solution:

$$|\lambda I_2 - A| = \begin{vmatrix} \lambda & -2 \\ -1 & \lambda - 1 \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2).$$

Thus the eignevalues of A are -1, 2. To find a -1-eigenvector:

$$-1 \cdot I_2 - A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix} \xrightarrow[-1 \cdot R_1 + R_2]{-1 \cdot R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix},$$

which shows that $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a -1-eigenvector of A.

Note: Any multiple of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is also a -1-eigenvector.

Example continued

To find a 2-eigenvector of A:

$$2I_2 - A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \xrightarrow{2 \cdot R_1 + R_2} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, or any multiple of it, is a 2-eigenvector for A.

The eigenvalues of B are the roots of

$$\begin{vmatrix} \lambda + 6 & -1 & -2 \\ 0 & \lambda - 3 & 4 \\ 0 & 0 & \lambda - 5 \end{vmatrix} = (\lambda + 6)(\lambda - 3)(\lambda - 5),$$

since B is upper triangular. Therefore, -6, 3, 5 are the eigenvalues of B.

Illustrative Examples

The following examples illustrate a number of different phenomena regarding the number of eigenvalues and eigenvectors.

Example 1. **No eigenvalues.** Consider the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then, $|\lambda I_2 - A| = \begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1$, which has no real roots, and thus A no eigenvalues (over $\mathbb{R}$). To see what is happening geometrically:
Let $\vec{i}$ and $\vec{j}$ denote the unit vectors along the x and y axes. Then:

$$A \cdot \vec{i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{j}$$

and

$$A \cdot \vec{j} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\vec{i}.$$

Thus, geometrically, the matrix A represents a rotation of the plane $\mathbb{R}^2$ counter-clockwise by 90 degrees.

Such a transformation does not preserve the direction of any vector in $\mathbb{R}^2$, so this explains why there are no eigenvalues or eigenvectors.

Example 2. An eigenvalue with multiplicity two and one eigenvector. Consider $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then

$$|\lambda I_2 - B| = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2,$$

so 1 is an eigenvalue of multiplicity two. That is, we think of the eigenvalues of B as as 1, 1.

To find the 1-eigenvectors:

$$1 \cdot I_2 - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},$$

which shows that the only eigenvectors are multiples of the basic vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Example 3. An eigenvalue with multiplicity two and two basic eigenvectors. Consider $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$$|\lambda \cdot I_3 - C| = \begin{vmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2(\lambda - 2),$$

so the eigenvalues are 1, 1, 2.

For the 1-eigenvectors:$1 \cdot I_3 - C =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{1 \cdot R_1 + R_2} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which gives rise to two independent eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

To elaborate: The 1-eigenvectors of C are the solutions to the homogeneous system of equations whose coefficient matrix is $1 \cdot I_3 - C$.

The RREF of this matrix is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. If we assume the variables are x, y, z, then the solutions are given by $x = r, y = 0, z = s$, where r, s can be any real numbers. In vector form, the solutions are:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

so the basic solutions to the homogeneous system are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and these are two independent 1-eigenvectors of C. It is easy to check that $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is a single independent 2-eigenvector of C.

Example 4. **A eigenvalue of multiplicity three and one basic eigenvector.** Consider $D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$$|\lambda \cdot I_3 - D| = \begin{vmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^3,$$

so 1 is an eigenvalue of D of multiplicity three, i.e., the eigenvalues of D are 1, 1, 1.

For the 1-eigenvectors of D:

$$1 \cdot I_3 - D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

from which it follows that all 1-eigenvectors of D are a multiple of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Example Revisited

For the matrix $A = \begin{bmatrix} 0 & 6 \\ 1 & 5 \end{bmatrix}$, we saw that the eigenvalues of A are 2 and 3, with associated eigenvectors $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Form a matrix P using the eigenvectors as columns, $P = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$. Then $P^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$. Moreover,

$$P^{-1}AP = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 6 & 6 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Thus, $P^{-1}AP$ is a diagonal matrix with the eigenvalues of A down its main diagonal.

In this case we say that A is **diagonalizable**.

Eigenvalues and eigenvectors play a central role in diagonalizing square matrices.

Class Example

Diagonalize the matrix $A = \begin{bmatrix} -1 & 3 \\ 0 & 5 \end{bmatrix}$. In other words find an invertible 2×2 matrix P such that $P^{-1}AP$ is a diagonal matrix with the eigenvalues of A down the diagonal.

Solution: First find the eigenvalues and corresponding eigenvectors of A.

$$|\lambda I_2 - A| = \begin{vmatrix} \lambda + 1 & -3 \\ 0 & \lambda - 5 \end{vmatrix} = (\lambda + 1)(\lambda - 5),$$

so the eigenvalues of A are -1 and 5.

For the -1-eigenvector:

$$-1 \cdot I_2 - A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 0 & -6 \end{bmatrix} \xrightarrow[6 \cdot R_1 + R_2]{-\frac{1}{3} \cdot R_1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which shows $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a basic -1-eigenvector for A.

Class Example continued

For the 5-eigenvector:

$$5I_2 - A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 0 & 0 \end{bmatrix},$$

which shows that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a basic 5-eigenvector for A. Putting the two eigenvectors as columns, we obtain $P = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Thus, $P^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$. Therefore, $P^{-1}AP =$

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} -1 & 5 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

Comment

Computing powers of a diagonalizable matrix: Suppose A is diagonalizable. We want to compute A^n, all n. Then $P^{-1}AP = D$, where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Note that $D^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$, for all n.

To compute the powers of A, we note that $A = PDP^{-1}$.

(i) $A^2 = PDP^{-1} \cdot PDP^{-1} = PD^2P^{-1}$.

(ii) $A^3 = A^2 \cdot A = PD^2P^{-1} \cdot PDP^{-1} = PD^3P^{-1}$.

(iii) Continuing, $A^n = PD^nP^{-1}$, for all n.

Thus, if A is diagonalizable, in order to calculate the powers of A, we just have to diagonalize A and compute the powers of a diagonal matrix.

Example

From an earlier example, for $A = \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix}$, A has eigenvalues $2, 3$ with eigenvectors $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Thus, we take $P = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$, and have $P^{-1}AP = D$, for $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. So

$$A^n = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^n \cdot \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \cdot 2^n & 2^{n+1} \\ -3^n & -3^n \end{bmatrix} = \begin{bmatrix} 3 \cdot 2^n - 2 \cdot 3^n & 2^{n+1} - 2 \cdot 3^n \\ -3 \cdot 2^n + 3^{n+1} & -2^{n+1} + 3^{n+1} \end{bmatrix}.$$