Lecture 10: Determinants, Inverses, and Eigenvalues

Definition

Recall the definitions needed to calculate the determinant of a square matrix, $A = (a_{ij})$.

(i) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the **determinant** of A is the real number $ad - bc$. We denote this number either by $\det(A)$ or $|A|$. For square matrices of larger size, the determinant is defined by reducing to matrices of smaller sizes.

(ii) Set A_{ij} to be the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of A.

(iii) Set $c_{ij}(A) = (-1)^{i+j} \cdot |A_{ij}|$. This is called the (i, j) **cofactor** of A.

(iv) Expansion along the *ith* row:

$$|A| = a_{i1} \cdot c_{i1}(A) + a_{i2} \cdot c_{i2}(A) + \cdots + a_{in} \cdot c_{in}(A).$$

(v) Expansion along the *jth* column:

$$|A| = a_{1j} \cdot c_{1j}(A) + a_{2j} \cdot c_{2j}(A) + \cdots + a_{nj} \cdot c_{nj}(A).$$

Example

Calculate the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 0 \\ 1 & -1 & -2 \end{bmatrix}$.

Expanding along the second row, we get

$$|A| = -3 \cdot \begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} - 0 \cdot |A_{23}|$$

$$= -3 \cdot (-4 + 3) + 2 \cdot (-2 - 3) = 3 + (-10) = -7.$$

Expanding along the third column, we get

$$|A| = 3 \cdot \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} - 0 \cdot |A_{32}| + (-2) \cdot \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}$$

$$= 3 \cdot (-3 - 2) + (-2) \cdot (2 - 6) = -15 + 8 = -7.$$

Properties of the Determinant

Let A be an $n \times n$ matrix.

(i) If A has a row or column of zeros, $|A| = 0$.

(ii) If two rows or two columns of A are the same, then $|A| = 0$.

(iii) If A' is obtained from A by multiplying a row (or column) of A by a number and adding it to a different row (or column), then $|A'| = |A|$.

(iv) If A' is obtained from A by interchanging two rows or two columns, then $|A'| = -|A|$.

(v) If A' is obtained from A by multiplying a row (or column) of A by $\lambda \neq 0$, then $|A'| = \lambda \cdot |A|$.

(vi) If A is **upper or lower triangular matrix**, i.e., all entries below the main diagonal or all entries above the main diagonal of A are zero, then $\det(A)$ is the product of the diagonal entries of A.

(vii) If B is an $n \times n$ matrix, then $|A \cdot B| = |A| \cdot |B| = |B \cdot A|$.

(viii) $|A^t| = |A|$.

Example

Use elementary rows operations to evaluate the determinant of
$A = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 1 & 2 \\ 4 & 4 & 4 \end{bmatrix}$.

Solution:

$$\begin{vmatrix} 2 & 4 & 6 \\ 2 & 1 & 2 \\ 4 & 4 & 4 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 4 & 4 & 4 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 4 & 4 & 4 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 0 & -4 & -8 \end{vmatrix}$$

$$= 2 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 0 & 0 & -\frac{8}{3} \end{vmatrix} = 2 \cdot (1 \cdot (-3) \cdot (-\frac{8}{3})) = 16.$$

CHECK:

$$|A| = 2 \begin{vmatrix} 1 & 2 \\ 4 & 4 \end{vmatrix} - 4 \begin{vmatrix} 2 & 2 \\ 4 & 4 \end{vmatrix} + 6 \begin{vmatrix} 2 & 1 \\ 4 & 4 \end{vmatrix} = 2 \cdot (-4) - 4 \cdot 0 + 6 \cdot 4 = 16.$$

Class Example

Use elementary row operations to put the matrix $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & 5 \\ -1 & 1 & 2 \end{bmatrix}$ into upper triangular form, and then find $\det(A)$.

Solution:

$$\begin{vmatrix} 1 & 3 & -1 \\ 2 & 4 & 5 \\ -1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 \\ 0 & -2 & 7 \\ -1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 \\ 0 & -2 & 7 \\ 0 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 \\ 0 & -2 & 7 \\ 0 & 0 & 15 \end{vmatrix} = -30.$$

Important Theorem

Theorem. Let A be an $n \times n$ matrix. The following statements are equivalent:

(i) The RREF of A is the $n \times n$ identity matrix.

(ii) The homogenous system $A \cdot X = 0$ has a unique solution.

(iii) A is an invertible matrix.

(iv) $\det(A) \neq 0$.

We've seen the role of the determinant in finding the inverse of a 2×2 matrix. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $|A| \neq 0$, $A^{-1} = \frac{1}{|A|} \cdot \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$.

For example, if $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, then $A^{-1} = \frac{1}{3} \cdot \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$.

A Formula for the Inverse in terms of the Determinant

Let $A = (a_{ij})$ be an $n \times n$ matrix and set $C = (c_{ij}(A))$, the cofactor matrix. .

(i) The **adjugate** or **classical adjoint** of A is the matrix $C^t = ((c_{ij}(A))^t$, denoted $\mathrm{adj}(A)$.

(ii) $A \cdot \mathrm{adj}(A) = |A| \cdot I_n = \mathrm{adj}(A) \cdot A$.

(iii) If $|A| \neq 0$, then $A^{-1} = \frac{1}{|A|} \cdot \mathrm{adj}(A)$.

Example

Suppose $A = \begin{bmatrix} -1 & 0 & 3 \\ 2 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix}$.

(i) For the cofactor matrix, we obtain

$$C = \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 3 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ 2 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1 & -2 & 6 \\ 9 & -1 & 3 \\ -6 & 7 & -2 \end{bmatrix}.$$

(ii) $\mathrm{adj}(A) = C^t = \begin{bmatrix} -1 & 9 & -6 \\ -2 & -1 & 7 \\ 6 & 3 & -2 \end{bmatrix}$.

Example continued

(iii) $A \cdot \mathrm{adj}(A) = \begin{bmatrix} -1 & 0 & 3 \\ 2 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 9 & -6 \\ -2 & -1 & 7 \\ 6 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 19 & 0 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 19 \end{bmatrix}$.

This tells us $|A| = 19$. Check:

$$|A| = -1 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = 1 + 18 = 19.$$

(iv) $A^{-1} = \frac{1}{|A|} \cdot \mathrm{adj}(A) = \frac{1}{19} \cdot \begin{bmatrix} -1 & 9 & -6 \\ -2 & -1 & 7 \\ 6 & 3 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{19} & \frac{9}{19} & -\frac{6}{19} \\ -\frac{2}{19} & -\frac{1}{19} & \frac{7}{19} \\ \frac{6}{19} & \frac{3}{19} & -\frac{2}{19} \end{bmatrix}$.

Cramer's Rule

Theorem. Let A be an invertible $n \times n$ matrix and let $A \cdot X = B$ be the matrix equation associated to a system of n equations in n unknowns. Let A_i denote the matrix obtained by replacing the i column of A with B. Then:

$$x_1 = \frac{|A_1|}{|A|}, x_2 = \frac{|A_2|}{|A|}, \ldots, x_n = \frac{|A_n|}{|A|}.$$

Example

Use Cramer's rule to find x_1 in the system of equations:

$$5x_1 + x_2 - x_3 = 4$$
$$9x_1 + x_2 - x_3 = 1$$
$$x_1 - x_2 + 5x_3 = 2$$

.

$$|A_1| = \begin{vmatrix} 4 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & -1 & 5 \end{vmatrix} = 4 \cdot \begin{vmatrix} 1 & -1 \\ -1 & 5 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 2 & 5 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 16 - 7 + 3 = 12.$$

$$|A| = \begin{vmatrix} 5 & 1 & -1 \\ 9 & 1 & -1 \\ 1 & -1 & 5 \end{vmatrix} = 5 \begin{vmatrix} 1 & -1 \\ -1 & 5 \end{vmatrix} - \begin{vmatrix} 9 & -1 \\ 1 & 5 \end{vmatrix} - \begin{vmatrix} 9 & 1 \\ 1 & -1 \end{vmatrix} = 20 - 46 + 10 = -16.$$

Thus, $x_1 = \frac{|A_1|}{|A|} = \frac{12}{-16} = -\frac{3}{4}$.

Class Example

For the example above, use Cramers Rule to find x_2.

Solution: $x_2 = \frac{|A_2|}{|A|}$,

$$|A_2| = \begin{vmatrix} 5 & 4 & -1 \\ 9 & 1 & -1 \\ 1 & 2 & 5 \end{vmatrix} = 5 \begin{vmatrix} 1 & -1 \\ 2 & 5 \end{vmatrix} - 4 \begin{vmatrix} 9 & -1 \\ 1 & 5 \end{vmatrix} - \begin{vmatrix} 9 & 1 \\ 1 & 2 \end{vmatrix} = 35 - 4 \cdot 46 - 17 = -166$$

From above: $|A| = -16$.Therefore, $x_2 = \frac{-166}{-16} = \frac{83}{8}$.

Eigenvectors and Eigenvalues for 2×2 matrices

Let A be a 2×2 matrix. The real number λ is called an **eigenvalue** of A if there exists a **non-zero** vector $v \in \mathbb{R}^2$ such that $Av = \lambda v$.

The vector v is called an eigenvector of A associated to λ or a λ-**eigenvector**.

Example: Let $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$, $v = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. Then

$$A \cdot v = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 4 \end{bmatrix} = 4 \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 4 \cdot v,$$

so 4 is an eigenvalue of A with eigenvector v.

Class Example

For the matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$, verify that -2 is an eigenvalue with associated eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Comment

If λ is an eigenvalue for the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with eigenvector v, then $Av = \lambda v$.

On the other hand, $\lambda v = (\lambda I_2) \cdot v = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \cdot v$

Thus, $Av = (\lambda I_2)v$, so $(\lambda I_2 - A)v = 0$.

Therefore, the homogeneous system of equations

$$(\lambda I_2 - A)X = \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix} \cdot X = 0$$

has a non-trivial solution.

Consequently: If λ is an eigenvalue of A with eigenvector v, then:

(i) $\det(\lambda I_2 - A) = \det(\begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix}) = 0$.

(ii) v is in the nullspace of $(\lambda I_2 - A)$.

Example Revisited

Find the eigenvalues of $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$.

To find the eigenvalues of $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$, we need λ such that $\det(\lambda I_2 - A)| = 0$. In other words, we want

$$\det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}) = \begin{vmatrix} \lambda - 3 & -5 \\ -1 & \lambda + 1 \end{vmatrix} = (\lambda - 3)(\lambda + 1) - 5 = 0.$$

So, we must solve the quadratic equation

$$(\lambda - 3)(\lambda + 1) - 5 = \lambda^2 - 2\lambda - 8 = 0.$$

Since $\lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2)$, we get $\lambda = 4$ or $\lambda = -2$, as expected.

Example Revisited

An eigenvector for 4 is in the nullspace of $(4I_2 - A)$ and an eigenvector for -2 is in the nullspace of $(-2I_2 - A)$.

If we apply EROs to $4I_2 - A$ we get

$$\begin{bmatrix} 4-3 & -5 \\ -1 & 4+1 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ -1 & 5 \end{bmatrix} \xrightarrow{-1\cdot R_1 + R_2} \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix},$$

from which we see that the nullspace of $4I_2 - A$ has $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ as a basic solution, which means that $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is a 4-eigenvector.

Similarly, $-2I_2 - A = \begin{bmatrix} -2-3 & -5 \\ -1 & -2+1 \end{bmatrix} = \begin{bmatrix} -5 & -5 \\ -1 & -1 \end{bmatrix}$ reduces to $\begin{bmatrix} -5 & -5 \\ 0 & 0 \end{bmatrix}$ which has nullspace generated by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, the expected eigenvector for -2.

Example

Find the eigenvectors and eigenvalues for $A = \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix}$.

Solution: Set $\det(\lambda I_2 - A) = \begin{vmatrix} \lambda - 0 & 6 \\ -1 & \lambda - 5 \end{vmatrix} = 0$. Thus,

$$\lambda(\lambda - 5) + 6 = \lambda^2 - 5\lambda + 6 = 0.$$

Therefore $\lambda = 2$ and $\lambda = 3$ are the eigenvalues of A.

Example continued

To find an eigenvector for 2 we need the nullspace of

$$(2I_2 - A) = \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}.$$

Using EROs:

$$\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} \xrightarrow{\frac{1}{2}\cdot R_2} \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix} \xrightarrow{-1\cdot R_1 + R_2} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix},$$

from which we see that the nullspace is generated by $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

In other words, $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is a 2-eigenvector of A.

In fact, all of the vectors $s \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, with $s \in \mathbb{R}$ are 2-eigenvectors of A.

Example continued

To find an eigenvector for 3 we need the nullspace of

$$(3I_2 - A) = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}.$$

Using EROs:

$$\begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix} \xrightarrow{\frac{1}{3}\cdot R_2} \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \xrightarrow{-1\cdot R_1 + R_2} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix},$$

from which we see that the nullspace is generated by $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

In other words, $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is a 3-eigenvector of A.

In fact, all of the vectors $r \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, with $3 \in \mathbb{R}$ are 2-eigenvectors of A.

Example continued

Where we are headed:. Form a 2×2 matrix using the eigenvectors as columns, $P = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$. Then $P^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$. Moreover,

$$P^{-1}AP = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 6 & 6 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Thus, $P^{-1}AP$ is a diagonal matrix with the eigenvalues of A down its main diagonal.

In this case we say that A is **diagonalizable**.

Eigenvalues and eigenvectors play a central role in diagonalizing square matrices.

Example

Here's an application: Suppose A is diagonalizable. Then $P^{-1}AP = D$, where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Thus, $A = PDP^{-1}$.

(i) $A^2 = PDP^{-1} \cdot PDP^{-1} = PD^2P^{-1}$.

(ii) $A^3 = A^2 \cdot A = PD^2P^{-1} \cdot PDP^{-1} = PD^3P^{-1}$.

(iii) Continuing, $A^n = PD^nP^{-1}$, for all n.

From the previous example, $A = \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix}$, $P = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. So

$$A^n = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^n \cdot \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \cdot 2^n & 2^{n+1} \\ -3^n & -3^n \end{bmatrix} = \begin{bmatrix} 3 \cdot 2^n - 2 \cdot 3^n & 2^{n+1} - 2 \cdot 3^n \\ -3 \cdot 2^n + 3^{n+1} & -2^{n+1} + 3^{n+1} \end{bmatrix}.$$