Exam 2 Review

0. Calculating determinants. Know various methods for calculating determinants. E.g., row or column expansion, and elementary row or column operations.

1. Eigenvalues, eigenvectors, and diagonalizability of square matrices. Let A be an $n \times n$ matrix.

(i) The real number λ is an **eigenvalue** of A is there exists a **non-zero** vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$. In this case, v is an **eigenvector** associate to λ.

(ii) The eigenvalues of A are the roots of $c_A(x)$, the *characteristic polynomial* of A. $c_A(x) = \det[xI_n - A]$.

(iii) For a given eigenvalue λ, the λ-eigenvectors are the non-zero zero vectors in the null space of the matrix $\lambda I_n - A$. The basic solutions in this null space are **basic** λ-eigenvectors and form a **basis** for the eigenspace E_λ.

(iv) If A is an $n \times n$ matrix, then, by definition, A is **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP = D$, where D is an $n \times n$ diagonal matrix.

(v) If A is diagonalizable, the diagonal entries of the matrix D in (iv) are the eigenvalues of A.

(vi) Suppose $c_A(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_r)^{e_r}$, then the eigenvalue λ_i has **multiplicity** e_i.

(vii) A is diagonalizable if and only if $c_A(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_r)^{e_r}$ and for each eigenvalue λ_i, e_i equals the dimension of E_{λ_i}.

In particular: A is diagonalizable if A has n distinct eigenvalues.

(viii) If A is diagonalizable, then the diagonalizing matrix P is obtained by taking the matrix whose columns are the collection of basic eigenvectors derived from A.

2. Applications of diagonalizability of square matrices. Suppose A is diagonalizable, with $P^{-1}AP = D$.

(i) $A = PDP^{-1}$, and therefore $A^n = PD^nP^{-1}$, for all $n \geq 1$.

(ii) For any square matrix B, e^B is the matrix given by the Taylor Series: $\sum_{n=0}^{\infty} \frac{1}{n!} B^n$.

(iii) If $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$, then $e^D = \text{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})$.

(iv) For A diagonalizable, $e^A = Pe^DP^{-1}$.

(v) Solving recurrence relations: A sequence of non-negative numbers $a_0, a_1, a_2, \ldots, a_k, \ldots,$ is called a **linear recursion sequence of length two** if there are fixed integers α, β, c, d such that:

(i) $a_0 = \alpha$.
(ii) $a_1 = \beta$.
(iii) $a_{k+2} = c \cdot a_k + d \cdot a_{k+1}$, for all $k \geq 0$.

To find a closed form solution for a_k, let $v_k = \begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix}$, and $A = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}$. Then $v_k = A^k \cdot v_0$, and a_k is the first coordinate of the vector v_k.

(vi) Solving systems of first order linear differential equations: Let $A = (a_{ij})$, be an $n \times n$ matrix. A system of first order linear differential equations is a system of equations of the form:

$$\begin{aligned}
x_1'(t) &= a_{11}x_1(t) + \cdots + a_{1n}x_n(t) \\
x_2'(t) &= a_{21}x_1(t) + \cdots + a_{2n}x_n(t) \\
\vdots \quad &= \qquad\qquad \vdots \\
x_n'(t) &= a_{n1}x_1(t) + \cdots + a_{nn}x_n(t),
\end{aligned}$$

where $x_i(t)$ is a real valued function of t. The numbers $x_1(0), \cdots, x_n(0)$ are called the *initial conditions* of the system.

Exam 2 Review

In matrix form, the system is given by the equation: $\mathsf{X}'(t) = A \cdot \mathsf{X}(t)$,

where $\mathsf{X}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ and $\mathsf{X}'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$.

The solution to the system is given by: $\mathsf{X}(t) = e^{At} \cdot \mathsf{X}(0)$.

3. Spanning sets, linear independence and bases in Euclidean space. Let $v_1, \ldots, v_r, w$ be columns vectors in $\mathbb{R}^n$. Let $A = [v_1\ v_2 \cdots\ v_r]$. Then:

(i) w belongs to span$\{v_1, \ldots, v_r\}$ if and only if the system of equations $A \cdot \mathrm{X} = w$ has a solution.

(ii) If $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$ is a solution to $A \cdot \mathrm{X} = w$, then $w = \lambda_1 v_1 + \cdots + \lambda_r v_r$.

(iii) $v_1, \ldots, v_r$ are linearly independent if and only if $A \cdot \mathrm{X} = \mathbf{0}$ has only the zero solution.

(iv) If $v_1, \ldots, v_r$ are **not** linearly independent and $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$ is a non-zero solution to $A \cdot \mathrm{X} = 0$, then

$$(*) \quad \lambda_1 v_1 + \cdots \lambda_r v_r = \mathbf{0}.$$

This means the vectors $v_1, \ldots, v_r$ are linearly dependent, and thus redundant.

(v) One can use (*) to write some v_i in terms of the remaining v's. Upon doing so:

$$\text{span}\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r\} = \text{span}\{v_1, \ldots, v_r\}.$$

(vi) One may continue to eliminate redundant vectors from among the v_i's.

As soon as one one arrives at a linearly independent subset of $v_1, \ldots, v_r$, this set of vectors forms a basis for the original subspace $\text{span}\{v_1, \ldots, v_r\}$.

The number of elements in the basis is then the dimension of $\text{span}\{v_1, \ldots, v_r\}$.

(vii) To test if the n vectors $v_1, \ldots, v_n$ in $\mathbb{R}^n$ are linearly independent, or span $\mathbb{R}^n$, or form a basis for $\mathbb{R}^n$, it suffices to show that $\det[v_1 \; v_2 \; \cdots \; v_n] \neq 0$.