April 8: Triple Integrals via Spherical and Cylindrical Coordinates

Example 1. Let's begin as we did with polar coordinates. We want a 3-dimensional analogue of integrating over a circle. So we integrate over B, the solid sphere of radius R to calculate its volume.

To calculate $\int\int\int_B \, dV$, we make the following substitutions:

(i) $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$

(ii) $dV = \rho^2\sin(\phi)d\rho, d\phi, d\theta, \;\; 0 \leq \rho \leq 1, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi.$

We get:

$$\begin{aligned}
\mathrm{vol}(B) &= \int\int\int_B \, dV \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2\sin(\phi)d\rho d\phi d\theta. \\
&= \frac{2\pi R^3}{3}\int_0^{\pi}\sin(\phi)\; d\phi \\
&= \frac{4\pi R^3}{3}.
\end{aligned}$$

Examples of Triple Integrals using Spherical Coordinates

One reason this works: The points described above lie on the sphere of radius ρ.

$$x^2 = \rho^2 \sin^2(\phi)\cos^2(\theta), \;\; y^2 = \rho^2 \; \sin^2(\phi)\sin^2(\theta), \; z^2 = \rho^2 \cos^2(\phi).$$

$$\begin{aligned} x^2 + y^2 + z^2 &= \rho^2\{\sin^2(\phi)\cos^2(\theta) + \sin^2(\phi)\sin^2(\theta) + \cos^2(\phi)\} \\ &= \rho^2\{sin^2(\phi)(\cos^2(\theta) + \sin^2(\theta)) + \cos^2(\phi)\} \\ &= \rho^2\{\sin^2(\phi) + \cos^2(\phi)\} \\ &= \rho^2. \end{aligned}$$

Later we will see the $\rho^2 \sin(\phi) d\rho d\phi d\theta$ is the volume of a spherical cube.

Which means we can replace dV in a Riemann sum by $\rho^2 \sin(\phi) d\rho d\phi d\theta$.

Examples of Triple Integrals using Spherical Coordinates

Here's the example from the end of Monday's class.

Example 2. Find the average value of $\sqrt{x^2+y^2+z^2}$ on the unit sphere B centered at the origin. We make the corresponding spherical substitutions.

Solution.

$$\text{Average Value} = \frac{1}{\text{vol}(B)} \int\int\int_B \sqrt{x^2+y^2+z^2}\, dV =$$

$$\frac{3}{4\pi} \int\int\int_B \sqrt{\rho^2 \sin^2(\phi)\cos^2(\theta) + \rho^2 \sin^2(\phi)\sin^2(\theta) + \rho^2\cos^2(\phi)}\rho^2 \sin(\phi) d\rho d\phi d\theta$$

$$= \frac{3}{4\pi} \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^3 \sin(\phi)\ d\rho d\phi d\theta$$

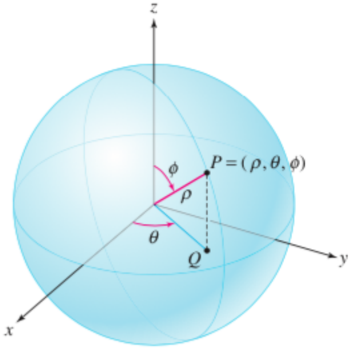
$$= \frac{3}{4\pi} \cdot \frac{2\pi}{4} \int_0^{\pi} \sin(\phi)\ d\phi$$

$$= \frac{3}{4\pi} \cdot \frac{2\pi}{4} \cdot 2 = \frac{3}{4}.$$

Spherical Coordinates

Every point in $\mathbb{R}^3$ lies on a sphere of radius ρ centered at (0,0,0) and thus can be expressed in terms of spherical coordinates.

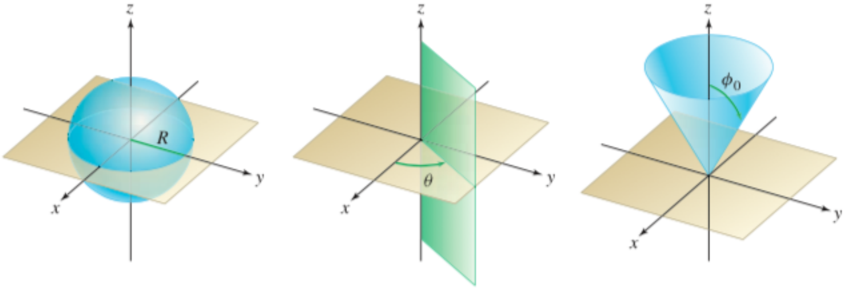
Here is typical point P using spherical coordinates



Note: $P = (\rho, \phi, \theta)$, with $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$.

Spherical Coordinates

If we set a spherical coordinate equal to a constant, we get:

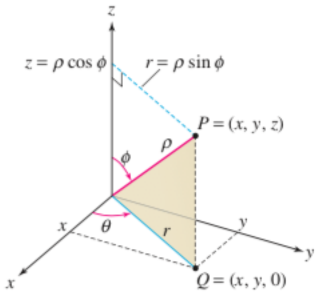


$\rho = R$
Sphere of radius R

$\theta = \theta_0$
Vertical half-plane

$\phi = \phi_0$
Right-circular cone

Here is the relation between the spherical coordinates and the rectangular coordinates of P.



Note:

$$x = r\cos(\theta) = \rho\sin(\phi)\cos(\theta), y = r\sin(\theta) = \rho\sin(\phi)\cos(\theta), z = \rho\cos(\phi).$$

Spherical Coordinates

Writing spherical coordinates in terms of rectangular coordinates.

$\rho = \sqrt{x^2 + y^2 + z^2}$

$\tan(\theta) = \frac{y}{x}$, so $\theta = \tan^{-1}(\frac{y}{x})$.

$\cos(\phi) = \frac{z}{\rho}$, so $\phi = \cos^{-1}(\frac{z}{\rho})$.

Spherical Coordinates

Class Example A. Find the rectangular coordinates of the point $P = (\rho, \phi, \theta) = (3, \frac{\pi}{3}, \frac{\pi}{4})$.

Solution

$$x = 3\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) = 3 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{6}}{4}$$

$$y = 3\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) = 3 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{6}}{4}$$

$$z = 3\cos\left(\frac{\pi}{3}\right) = 3 \cdot \frac{1}{2} = \frac{3}{2}.$$

Spherical Coordinates

Example 3. Find the spherical coordinates of the point $P = (x, y, z) = (-1, 1, \sqrt{6})$.

Solution. $\rho = \sqrt{(-1)^2 + 1^2 + (\sqrt{6})^2} = \sqrt{8} = 2\sqrt{2}$.

From $z = \rho \cos(\phi)$ we have $\sqrt{6} = 2\sqrt{2}\cos(\phi)$. Thus:

$\cos(\phi) = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2}$. $\phi = \frac{\pi}{6}$.

$\theta = \tan^{-1}(\frac{-1}{1}) = \tan^{-1}(-1) = \frac{3\pi}{4}$, since $(x, y) = (-1, 1)$.
Thus, in spherical coordinates, $P = (2\sqrt{2}, \frac{\pi}{6}, \frac{3\pi}{4})$.

Spherical Coordinates

Fubini's Theorem for Triple Integrals Using Spherical Coordinates

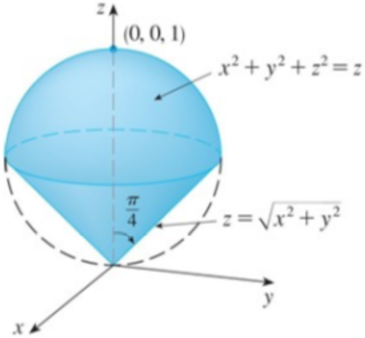
THEOREM 3 Triple Integrals in Spherical Coordinates For a region $\mathcal{W}$ defined by

$$\theta_1 \leq \theta \leq \theta_2, \qquad \phi_1 \leq \phi \leq \phi_2, \qquad \rho_1(\theta, \phi) \leq \rho \leq \rho_2(\theta, \phi)$$

the triple integral $\iiint_{\mathcal{W}} f(x, y, z)\, dV$ is equal to

$$\int_{\theta_1}^{\theta_2} \int_{\phi=\phi_1}^{\phi_2} \int_{\rho=\rho_1(\theta,\phi)}^{\rho_2(\theta,\phi)} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta \qquad \boxed{8}$$

Example 4. Calculate the volume of the solid B bounded by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $z = z^2 + y^2 + z^2$.



For the sphere:

$$\rho \cos(\phi) = z = x^2 + y^2 + x^2 = \rho^2,$$

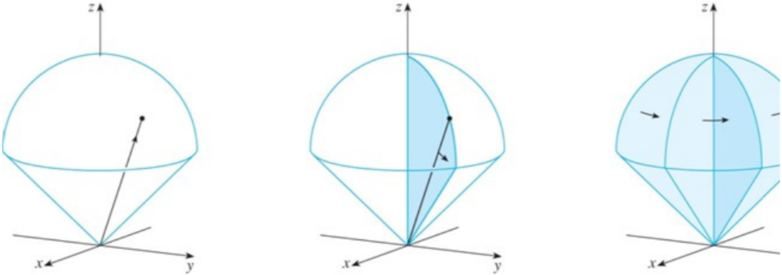
So $\rho = \cos(\phi)$.

The cone:

$$\rho\cos(\phi) = \sqrt{(\rho\sin(\phi)\cos(\theta))^2 + (\rho\sin(\phi)\sin(\theta))^2}$$
$$= \rho\sin(\phi),$$

Where the cone and sphere meet: $\rho\cos(\phi) = \rho\sin(\phi)$ so: $\cos(\phi) = \sin(\phi)$.

Therefore: $\phi = \frac{\pi}{4}$. θ varies from 0 to 2π.



ρ varies from 0 to $\cos\phi$ while ϕ and θ are constant.

ϕ varies from 0 to $\pi/4$ while θ is constant.

θ varies from 0 to 2π.

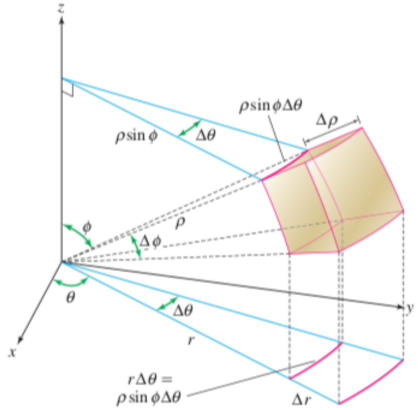
Examples of Triple Integrals using Spherical Coordinates

$$
\begin{aligned}
\mathrm{vol}(B) &= \int\int\int_B dV \\
&= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos(\phi)} \rho^2 \sin(\phi) d\rho d\phi d\theta \\
&= \frac{1}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \cos^3(\phi)\sin(\phi)\ d\phi d\theta \\
&= \frac{1}{3}\int_0^{2\pi} -\frac{\cos^4(\phi)}{4}\Big|_0^{\phi=\frac{\pi}{4}}\ d\theta \\
&= \frac{1}{12}\int_0^{2\pi}\{-(\frac{\sqrt{2}}{2})^4 + 1\}\ d\theta \\
&= \frac{2\pi}{12}\{-(\frac{\sqrt{2}}{2})^4 + 1\} \\
&= \frac{\pi}{8}.
\end{aligned}
$$

Spherical Coordinates

Why is $dV \approx \rho^2 \sin(\phi)\ d\rho d\phi d\theta$?

We can use the small spherical wedge in the Riemann sums:



Recall that $r = \rho \sin(\phi)$. The missing edge is approximately $\rho\Delta\phi$. The volume of the wedge is approximately:

$$(\rho \sin(\phi)\Delta\theta) \cdot (\Delta\rho) \cdot (\rho\Delta\phi) = \rho^2 \sin(\phi)\Delta\rho\Delta\phi\Delta\theta.$$

Examples of Triple Integrals using Spherical Coordinates

Example 5. Calculate $\int\int\int_B e^{-\sqrt{x^2+y^2+z^2}}\ dV$, where B is that portion of the solid sphere of radius R in the first octant.

In polar coordinates, B is determined by the inequalities: $0 \leq \rho \leq R$, $0 \leq \phi \leq \frac{\pi}{2}$, $0 \leq \theta \leq \frac{\pi}{2}$.

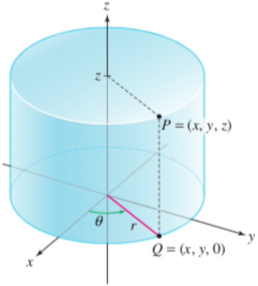
$$\begin{aligned}
\int\int\int_B e^{-\sqrt{x^2+y^2+z^2}}\ dV &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^R e^{-\rho}\rho^2\sin(\phi)\ d\rho d\phi\ d\theta \\
&= \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\int_0^R e^{-\rho}\rho^2\sin(\phi)\ d\rho d\phi \\
&= \frac{\pi}{2}\int_0^R e^{-\rho}\rho^2\ d\rho \\
&= \frac{\pi}{2}\{(-\rho^2-2\rho-2)e^{\rho}\}_0^R \\
&= \frac{\pi}{2}\{(-R^2-2R-2)e^{-R}+2\}
\end{aligned}$$

Cylindrical Coordinates

Cylindrical coordinates are essentially like polar coordinates, though with the extra variable z.

To transform a triple integral into cylindrical coordinates, we set:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z, \quad dV = r dz dr d\theta.$$

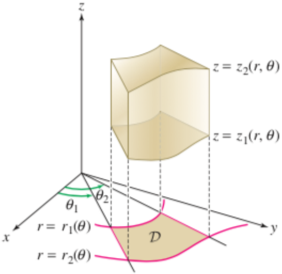


In cylindrical coordinates $P = (r, \theta, z)$.

Cylindrical Coordinates

We can easily guess what the volume of a cylindrical wedge should be.

Solution: It should be approximately the area of the corresponding polar rectangle times change in the z direction (height).



$$dV \approx \mathrm{area}(D) \cdot \Delta z = (r \Delta r \Delta \theta) \cdot \Delta z = r dr d\theta dz.$$

Cylindrical Coordinates

Here is a version of Fubini's Theorem for cylindrical coordinates.

THEOREM 2 Triple Integrals in Cylindrical Coordinates For a continuous functi
f on the region

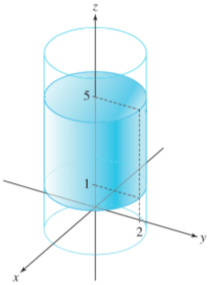
$$\theta_1 \le \theta \le \theta_2, \qquad r_1(\theta) \le r \le r_2(\theta), \qquad z_1(r,\theta) \le z \le z_2(r,\theta),$$

the triple integral $\iiint_{\mathcal{W}} f(x,y,z)\,dV$ is equal to

$$\int_{\theta_1}^{\theta_2} \int_{r=r_1(\theta)}^{r_2(\theta)} \int_{z=z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Examples of Triple Integrals using Cylindircal Coordinates

Example 6. Integrate $z\sqrt{x^2 + y^2}$ over the cylinder B:

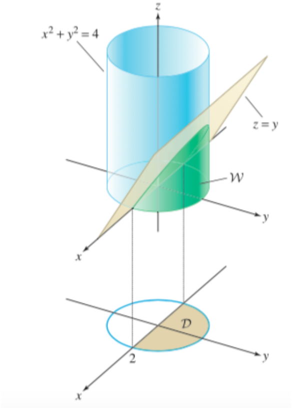


In polar coordinates $B : 0 \le r \le 2, 0 \le \theta \le 2\pi, 1 \le z \le 5$.

Examples of Triple Integrals using Cylindircal Coordinates

$$
\begin{aligned}
\int\int\int_B z\sqrt{x^2+y^2} &= \int_0^{2\pi}\int_0^2\int_1^5 z\sqrt{(r\cos(\theta))^2+(r\sin(\theta))^2}\ r\ dzdrd\theta \\
&= \int_1^5\int_0^{2\pi}\int_0^2 zr^2\ drd\theta dz \\
&= 2\pi\int_1^5\int_0^2 zr^2\ drdz \\
&= 2\pi\int_1^5 z(\frac{2^3}{3}-0)\ dz \\
&= \frac{16\pi}{3}\int_1^5 z\ dz \\
&= \frac{16\pi}{3}\{\frac{5^2}{2}-\frac{1}{2}\} \\
&= 64\pi.
\end{aligned}
$$

Example 7. Calculate $\int\int\int_W z\, dV$ for W: $0 \leq x^2 + y^2 \leq w$ and $0 \leq y \leq z$.



To describe W is cylindrical coordinates: $0 \leq z \leq y$, so

$$0 \leq z \leq r \sin(\theta).$$

Examples of Triple Integrals using Cylindircal Coordinates

Since $y \geq 0$, the project of W onto the xy-plane is the semi-circle D. Thus, $0 \leq r \leq 2$ and $0 \leq \theta \leq \pi$.

$$\int\int\int_W z\ dV = \int_0^{\pi}\int_0^2\int_0^{r\sin(\theta)} z\ rdzdrd\theta$$
$$= \frac{1}{2}\int_0^{\pi}\int_0^2 r^3\sin^2(\theta)\ drd\theta$$
$$= \frac{1}{2}\int_0^{\pi}\frac{16}{4}\sin^2(\theta)\ d\theta$$
$$= 2\int_0^{\pi}\frac{1}{2}-\frac{1}{2}\cos(2\theta)\ d\theta$$
$$= 2\{\frac{\theta}{2}-\frac{1}{4}\sin(\theta)\}_0^{\pi}$$
$$= 2\pi.$$