SPRING 2021: MATH 147 QUIZ 2 SOLUTIONS

Each question is worth 5 points. You must justify your answer to receive full credit.

1. Find the equation of the plane tangent to the graph of $z = x^2 + 2y^2 + 1$ at the point (1,1,4).

Solution. $f_x(x, y) = 2x$, so $f_x(1, 1) = 2$. $f_y(x, y) = 4y$, so $f_y(1, 1) = 4$. f(1,1) = 4. The equation of the tangent plane is

$$z = 2(x - 1) + 4(y - 1) + 4.$$

2. Use the limit definition to find directional derivative of $f(x, y) = 3x^2 + 2xy + 4$ at (1,1) in the direction of the vector $a\vec{i} + b\vec{j}$.

Solution. We set $\vec{u} = \frac{a}{\sqrt{a^2+b^2}}\vec{i} + \frac{b}{\sqrt{a^2+b^2}}\vec{j}$, the unit vector in the direction of $a\vec{i} + b\vec{j}$. For ease of notation, set $A := \frac{a}{\sqrt{a^2+b^2}}$ and $B := \frac{b}{\sqrt{a^2+b^2}}$. Then,

$$\begin{aligned}
D_{\vec{u}}f(1, 1) &= \lim_{h\to 0} \frac{f(1 + hA, 1 + hB) - f(1, 1)}{h} \\
&= \lim_{h\to 0} \frac{(3(1 + hA)^2 + 2(1 + hA)(1 + hB) + 4) - 9}{h} \\
&= \lim_{h\to 0} \frac{3 + 6hA + 3h^2A^2 + 2 + 2hA + 2hB + 2h^2AB + 4 - 9}{h} \\
&= \lim_{h\to 0} 8A + 2B + 3hA^2 + 2hAB \\
&= 8A + 2B = \frac{8a + 2b}{\sqrt{a^2 + b^2}}.
\end{aligned}$$

3. For $f(x, y) = -x^2y + xy^2 + xy$ and the point $P = (2, 1)$: (a) Find the direction of maximal rate of increase of $f(x, y)$ at P; (b) What is the maximum value of $D_{\vec{u}}f(P)$?; Give a direction $\vec{u}$ such that $D_{\vec{u}}f(P) = 0$.

Solution. For (a), the direction is $\nabla f(2, 1)$. For this, $\nabla f(x, y) = (-2xy + y^2 + y)\vec{i} + (-x^2 + 2xy + x)\vec{j}$, so $\nabla f(2, 1) = -2\vec{i} + 2\vec{j}$.

For (b), we take $|\nabla f(2, 1)| = | - 2\vec{i} + 2j| = \sqrt{8}$.

For (c), we want $\vec{u}$ such that $\vec{u} \cdot \nabla f(2, 1) = 0$, i.e., $\vec{u} \cdot (-2\vec{i} + 2\vec{j}) = 0$. The vector $\vec{v} = 2\vec{i} + 2\vec{j}$ is orthogonal to $-2\vec{i} + 2\vec{j}$, so we take $\vec{u} = \frac{2}{\sqrt{8}}\vec{i} + \frac{2}{\sqrt{8}}\vec{j}$.